Limits & Derivatives
MAΘ State 2008

(1) $f(x+\Delta x) = f(x) + f'(x)\Delta x$

$f(83) = \sqrt{81} + \frac{1}{2\sqrt{81}} \cdot 2$

$f(83) = 9 + \frac{1}{9} = \frac{82}{9}$ A

(2) $\lim_{x\to 0} \frac{x^3 - 6x^2 + x - \sin x}{x^2} = \frac{0}{0}$ Use L'hopital's rule

$\lim_{x\to 0} \frac{3x^2 - 12x + 1 - \cos x}{2x} = \frac{0}{0}$ Use L'hopital again

$\lim_{x\to 0} \frac{6x - 12 + \sin x}{2} = \frac{-12}{2} = -6$ C

(3) $\lim_{h\to 0} \frac{f(x+2h) - f(x-2h)}{2h} = 2f'(x)$

Multiply by $\frac{2}{2}$ to make it the definition of a derivative

$f'(x) = 4x^3 + 9x^2 - 2x$

$2f'(x) = 8x^3 + 18x^2 - 4x$ E

(4) $f(x) = x^3 - \sin x$

$f'(x) = 3x^2 - \cos x$

$f''(x) = 6x + \sin x$ C

(5) $\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x = y \rightarrow \lim_{x\to\infty} \ln\left(1+\frac{2}{x}\right)^x = \ln y$

$\lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$ Use L'hopital

$\lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x}} \cdot \frac{-2}{x^2}}{-\frac{1}{x^2}} = \ln y \rightarrow \lim_{x\to\infty} \frac{2}{1+\frac{2}{x}} = \ln y$

5 cont. $z = \ln y \quad e^z = y \quad e^2 = \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$ D

6 $x^2 + \ln x - \sin^2 x \rightarrow \frac{dy}{dx} = 2x + \frac{1}{x} - 2\sin x\cos x$

$2\sin x\cos x = \sin 2x$

$\frac{dy}{dx} = 2x + \frac{1}{x} - \sin 2x$ C

7 $xy^2 - 2y = \frac{y}{x} + \sin y$ Use implicit differentiation

$y^2 + 2xyy' - 2y' = \frac{y'}{x} - \frac{y}{x^2} + \cos y\, y'$

$y^2 + \frac{y}{x^2} = \cos y\, y' + 2y' - 2xyy' + \frac{y'}{x} \quad y' = \frac{dy}{dx}$

$$\frac{y^2 + \frac{y}{x^2}}{\cos y + 2 - 2xy + \frac{1}{x}} = \frac{dy}{dx}$$ A

8 $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$\frac{dy}{dx} = \frac{2t}{\cos t} \qquad \frac{d^2y}{dx^2} = \frac{\frac{2\cos t + 2t\sin t}{\cos^2 t}}{\cos t}$

$\frac{d^2y}{dx^2} = \frac{2\cos t + 2t\sin t}{\cos^3 t} \qquad t = \pi$

$\frac{2(-1) + 2(\pi)(0)}{(-1)^3} = \frac{-2}{-1} = 2$ A

⑨ $\lim_{x\to 2}\left(\frac{x^3-x^2-4x+4}{x-2}\right) = \frac{0}{0}$ factor by grouping

$\lim_{x\to 2}\left(\frac{x^2(x-1)-4(x-1)}{(x-2)}\right) \to \lim_{x\to 2}\left(\frac{(x^2-4)(x-1)}{(x-2)}\right)$

$\lim_{x\to 2}\left(\frac{\cancel{(x-2)}(x+2)(x-1)}{\cancel{(x-2)}}\right) \to \lim_{x\to 2}\left((x+2)(x-1)\right) = 4$ (B)

⑩ $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

$f^{-1}(2)=3 \qquad f'(3)=5 \quad F'(2)=\frac{1}{5}$ (A)

⑪ concavity → use 2nd derivative

$f(x)=x^4+5x^3+6x^2+3x-9$
$f'(x)=4x^3+15x^2+12x+3$
$f''(x)=12x^2+30x+12 = 6(2x^2+5x+2) = 6(2x+1)(x+2)$

++ | −−− | ++ test zones in 2nd derivative
$-2 \quad -\frac{1}{2}$

$(-\infty,-2)\cup(-\frac{1}{2},\infty)$ (D)

⑫ $x=-3$ is not in the domain. ~~$\log_2 3$~~ (E)

⑬ $y=x^2-4x-5 \quad f'(x)=2x-4 \quad f'(1)=-2$

$y+8=-2(x-1)$ (B)

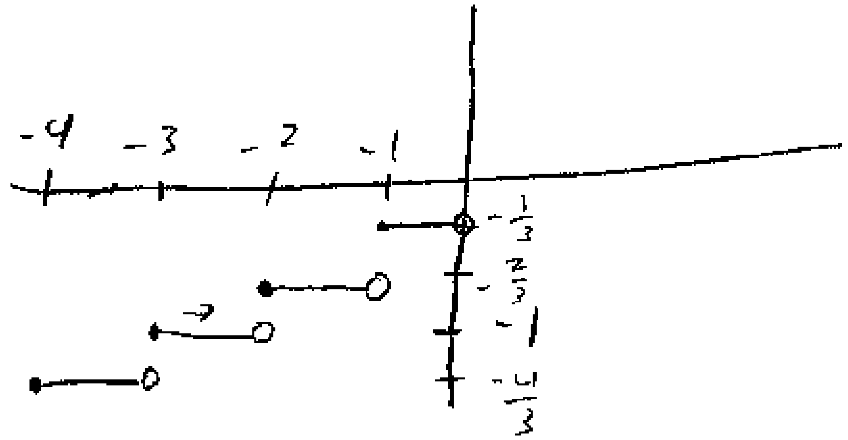
⑭ Increasing/Decreasing use 1st derivative

$f'(x)=x^2-x-6=(x-3)(x+2)$ +++ | −−− | +++
$-2 \quad 3$

(C) $(-\infty,-2)\cup(3,\infty)$

(15) Draw a graph $f(x) = \frac{\lceil x \rceil}{3}$

$\lim\limits_{x \to -2^-} \frac{\lceil x \rceil}{3} = -1$



A

(16) Use the chain rule

$f(x) = \sin^2(x^2-4)$

C

$f'(x) = 2\sin(x^2-4)\cos(x^2-4) \cdot 2x = 4x\sin(x^2-4)\cos(x^2-4)$

(17) for continuity and differentiability the function and its derivative must be continuous.

$2x-4 = Ax^2 - Bx + 4$ and $2 = 2Ax - B$

when $x = 4$

$2 = 8A - B$

$4 = 16A - 4B + 4$

$0 = 4A - B$

$-2 = -8A + B$

$-2 = -4A$

$\frac{1}{2} = A$

$2 = B$

$A(A+B) = \frac{1}{2}\left(\frac{5}{2}\right) = \frac{5}{4}$ E

(18) The sumation becomes the integral from 0 to 1.

$\frac{i}{n} = x$ $\quad \frac{1}{n} = dx$ $\quad \sum\limits_{i=1}^{\infty}\left(\frac{n^2+i^2}{n^3}\right) = \sum\limits_{i=1}^{\infty}\left(1+\frac{i^2}{n^2}\right)\frac{1}{n} = \int_0^1 (1+x^2)\,dx$

$\sum\limits_{i=1}^{\infty}\left(\frac{n^2+i^2}{n^3}\right) = \frac{4}{3}$ B

(19) The derivative of $\arcsin x$ is $\frac{dx}{\sqrt{1-x^2}}$

substitute x^2 for x

$$\frac{2x}{\sqrt{1-x^2}}$$ B

(20) Use chain rule

$f(x) = \sin(\arccos(x))$ $\quad f'(s) = \cos(\arccos(s)) \cdot \frac{d}{ds}(\arccos(s))$

C $\quad f'(s) = s \cdot \frac{-1}{\sqrt{1-s^2}} = \frac{-s}{\sqrt{1-s^2}}$

(21) $f(x)$ has local maxima, but no absolute maximum

The maximum is ∞. E

(22) Mean value theorem $\quad \frac{F(b)-F(a)}{b-a} = F'(c)\ [a,b]$

$f(4) = 54 \quad f(-1) = 4$

$$\frac{54-4}{4--1} = 3x^2-3 \qquad 10 = 3x^2 - 3$$

A $\quad 13 = 3x^2$

$\frac{\sqrt{39}}{3} = x$ drop the negative, it is not on the interval

(23) derivative of the velocity function is acceleration

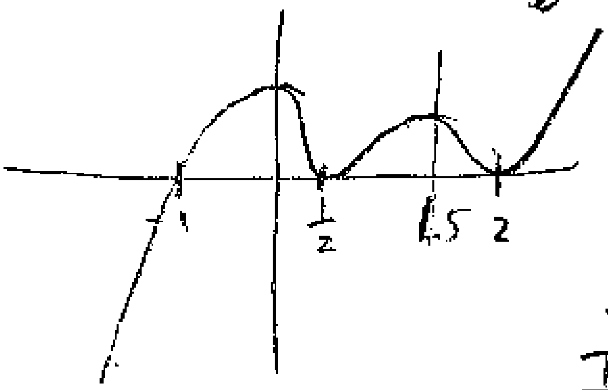
$f'(x) = \frac{3}{2}x^2 + 4x + 5 \quad f'(4) = 24 + 16 + 5 = 45$ A

(24) $\int_1^\infty \frac{1}{x^2}dx = \lim_{a\to\infty} \int_1^a \frac{1}{x^2} = \lim_{a\to\infty} \left[\frac{-1}{x}\right|_1^a$

$\lim_{a\to\infty}\left[-\frac{1}{a} - -1\right] = 0+1 = 1$ (C)

(25) 5^{th} degree polynomial enters in 3^{rd} quadrant and exits in 1^{st} quadrant zeros at $-1, \frac{1}{2}, 2$

$\frac{1}{2}$ → double, 2 → double

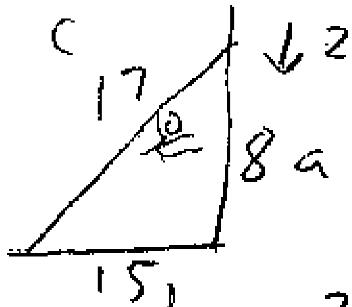


I True

II False use 2^{nd} derivative

III True (C)

(26)



$a^2+b^2=c^2$ take derivative

$2aa' + 2bb' = 2cc'$ $\quad c' = 0$

$2\cdot 8\cdot(-2) + 2\cdot 15\cdot b' = 0$

$30b' = 32$ (C)

$b' = \frac{16}{15}$

(27) The difference between the "h's" must be the denominator.

(D)

(28) Only the leading coefficients matter

$\lim_{x\to\infty}\frac{x^2}{\sqrt{5x^4}} = \lim_{x\to\infty}\frac{x^2}{\sqrt{5}x^2} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$ (D)

(29) $\ln|\sec x \tan x|$ use chain rule

$$\frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x) = \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}$$

$$= \sec x$$ (B)

(30) sin x oscillates, the limit DNE.

(D)