National MAΘ 2008 – Mu Limits and Derivatives Answers

1) D
2) C
3) D
4) A
5) B
6) D
7) E
8) C
9) C
10) B
11) B
12) B
13) E
14) C
15) D
16) A
17) B
18) A
19) A
20) C
21) C
22) E
23) B
24) D
25) C
26) A
27) B
28) B
29) C
30) D

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1) $\lim_{x\to\infty}\dfrac{9x^3-3x^2+7x-2007}{3x^3+6x+2007}=$

$$\lim_{x\to\infty}\frac{9-\frac{3}{x}+\frac{7}{x^2}-\frac{2007}{x^3}}{3+\frac{6}{x^2}+\frac{2007}{x^3}}=\frac{9-0+0-0}{3+0+0}=3 \text{ \textbf{D}}$$

2) $\lim_{t\to 0}\dfrac{f(t+1)-f(1)}{t}=f'(1)=18(1)=18$ **C**

3) $y'=\dfrac{4x^2\exp(x^2)-2\exp(x^2)}{4x^2}=\dfrac{\exp(x^2)(2x^2-1)}{2x^2}$

$$y''=\exp(x^2)\frac{2x^2(4x)-4x(2x^2-1)}{4x^4}+\ldots$$

$$+2x\exp(x^2)\frac{2x^2-1}{2x^2}=\frac{\exp(x^2)(2x^4-x^2+1)}{x^3} \text{ \textbf{D}}$$

4) $A=\dfrac{1}{2}x(2x-x^2)$ $\qquad \dfrac{dA}{dx}=2x-\dfrac{3}{2}x^2=0$

$x=0$ (minimum area) or $x=\dfrac{4}{3}$ (maximum area)

$$A_{\max}=\frac{1}{2}\left(\frac{4}{3}\right)\left(2\left(\frac{4}{3}\right)-\left(\frac{4}{3}\right)^2\right)=\frac{16}{27} \text{ \textbf{A}}$$

5) $h(x)=\sqrt{x+h(x)}$ $\qquad h(2)=\sqrt{2+h(2)}\Rightarrow h(2)=2$

$$h'(x)=\frac{1+h'(x)}{2\sqrt{x+h(x)}}\Rightarrow h'(2)=\frac{1+h'(2)}{2\sqrt{2+2}};\ h'(2)=\frac{1}{3} \text{ \textbf{B}}$$

6) $\lim_{x\to 1}h(x)$ exists if $\lim_{x\to 1^+}h(x)=\lim_{x\to 1^-}h(x)$

$\lim_{x\to 1^+}h(x)=\lim_{x\to 1^+}x+7=8$

$\lim_{x\to 1^-}h(x)=\lim_{x\to 1^+}9-x^2=8$

Since $\lim_{x\to 1^+}h(x)=\lim_{x\to 1^-}h(x)$, the limit exists regardless of the value of *a*. **D**

7) Take derivative with respect to *x* on both sides

$$12x^2-2xy^2-2x^2y\frac{dy}{dx}+4y\frac{dy}{dx}=0$$

$$\left.\frac{dy}{dx}\right|_{(-1,2)}=\left.\frac{2xy^2-12x^2}{4y-2x^2y}\right|_{(-1,2)}=-5 \text{ \textbf{E}}$$

8) $y'=3x^2-4x+1$ $\qquad y'>0$ for all $x\in(1,4)$

y is increasing on the interval $(1,4)$

$y''=6x-4$ $\qquad y''>0$ for all $x\in(1,4)$

y is concave up on the interval $(1,4)$ **C**

9) If $a\ge 1$, $a^n\to\infty$ AND $n^a\to\infty$ divergent

If $0\le a<1$ then $\lim_{n\to\infty}\dfrac{n^a}{a^{-n}}\equiv\lim_{n\to\infty}\dfrac{a\cdot n^{a-1}}{-\ln(a)a^{-n}}=0$

If $-1<a<0$, let $b=-a$

$$\lim_{n\to\infty}(-b)^n n^{-b}=\lim_{n\to\infty}\frac{(-1)^n b^n}{n^b}=0.$$

If $a=-1$, $\lim_{n\to\infty}(-1)^n n^{-1}=\lim_{n\to\infty}\dfrac{(-1)^n}{n}=0$.

If $a<-1$, let $b=-a$:

$\lim_{n\to\infty}(-b)^n n^{-b}=\lim_{n\to\infty}\dfrac{(-1)^n b^n}{n^b}=\infty$ because b^n diverges more rapidly than *n* raised to any power.

Thus the limit converges only when $-1\le a<1$. **C**

10) $r=\sqrt{(x-2)^2+(y+5)^2}=\sqrt{(y^2+1)^2+(y+5)^2}$

$$\frac{dr}{dy}=\frac{2(y^2+1)(2y)+2(y+5)}{2\sqrt{(y^2+2)^2+(y+5)^2}}=0$$

Set numerator to zero: $4y^3+6y+10=0$

$y=-1\Rightarrow r=\sqrt{20}=2\sqrt{5}$ **B**

11) $V=\dfrac{4\pi}{3}r^3$ $\qquad \dfrac{dV}{dt}=4\pi r^2\dfrac{dr}{dt}=4\pi$

$$\left.\frac{dV}{dt}\right|_{r=2}=4\pi(2)^2\frac{dr}{dt}=4\pi\Rightarrow\frac{dr}{dt}=\frac{1}{4}$$

$A=4\pi r^2$ $\qquad \dfrac{dA}{dt}=8\pi r\dfrac{dr}{dt}=8\pi(2)\left(\dfrac{1}{4}\right)=4\pi$ **B**

12) $\lim_{n\to\infty}\left(1+\dfrac{r}{n}\right)^{tn+m}=e^{rt}$ $\quad \lim_{n\to\infty}\left(1-\dfrac{1}{n}\right)^{2n-3}=e^{-2}$ **B**

13) $\ln h(x)=\dfrac{1}{x}\ln x$ $\qquad \dfrac{h'(x)}{h(x)}=\dfrac{1}{x^2}-\dfrac{1}{x^2}\ln x$

$h'(x)=\left(\dfrac{1}{x^2}-\dfrac{1}{x^2}\ln x\right)x^{1/x}=0$, $\quad x^{1/x}\ne 0$ for any *x*

$\dfrac{1}{x^2}-\dfrac{1}{x^2}\ln x=0\Rightarrow x=e$ $\qquad$ Max = $e^{1/e}$ **E**

14) First derivative: $n!x^{n!-1}$

Second derivative: $n!(n!-1)x^{n!-2}$

Third derivative: $n!(n!-1)(n!-2)x^{n!-3}$

n^{th} derivative:

$n!(n!-1)(n!-2)\cdot\ldots\cdot(n!-n+2)(n!-n+1)x^{n!-n}$

The coefficient equals $\dfrac{(n!)!}{(n!-n)!}$ **C**

15) $\left|\left(\dfrac{x}{3}+\pi-1\right)-\pi\right|<0.3$ $\qquad \dfrac{1}{3}|x-3|<0.3$

$|x-3|<0.9$ Choose $\delta = 0.9$ **D**

16) $\dfrac{dy}{dx} = \dfrac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$ $r(\theta) = 1-\cos(2\theta)$ $r'(\theta) = 2\sin(2\theta)$

$$\frac{dy}{dx} = \frac{\left(\sqrt{3}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{\left(\sqrt{3}\right)\left(\frac{1}{2}\right)-\left(\frac{3}{2}\right)\left(\frac{\sqrt{3}}{2}\right)} = -3\sqrt{3}\ \textbf{A}$$

17) I) $g'(x)$ could be negative in between the integer values – False
II) There is a critical point at $x=1$ but not necessarily a minimum – False
III) $g'(0)=0$ – True
IV) Same reason as I – False **B**

18) $\lim_{n\to\infty}\frac{1}{n}\left[e^{1/n}+e^{2/n}+\ldots+e^{n/n}\right] = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}e^{k/n}$

This is a Riemann Integral approximation of

$\int_0^1 e^x dx = e-1$ **A**

19) $g'(x) = 2(1+f(x))f'(x)$

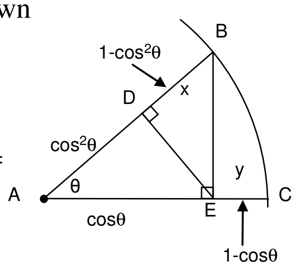
$g''(x) = 2(1+f(x))f''(x) + 2[f'(x)]^2$

$g''(0) = 2(1+1)(-1)+2[0]^2 = -4$ **A**

20) $F'(x) = \sin(x^2)$ $F''(x) = 2x\cos(x^2)$

A) $F''(x)<0$ concave down
B) $F''(x)<0$ concave down
C) $F''(x)>0$ concave up **C**
D) $F''(x)<0$ concave down

21) $x = 1-\cos^2\theta$
$y = 1-\cos\theta$



$\lim_{\theta\to 0}\frac{x}{y} = \lim_{\theta\to 0}\frac{1-\cos^2\theta}{1-\cos\theta} =$

$\lim_{\theta\to 0}(1+\cos\theta) = 2$ **C**

22) $\lim_{n\to\infty}\left(\frac{1}{5}\right)^n + \lim_{n\to\infty}\left(\frac{2}{5}\right)^n + \lim_{n\to\infty}\left(\frac{3}{5}\right)^n + \lim_{n\to\infty}\left(\frac{4}{5}\right)^n$

$=0+0+0+0=0$ **E**

23) $f'(c) = \dfrac{f(1)-f(-2)}{1-(-2)} = 2$, $3c^2-1=2 \Rightarrow c=\pm 1$

c cannot equal 1 since it's an endpoint. Sum of the values of c is –1. **B**

24) $\lim_{x\to -1}\dfrac{2x^2-3x+1}{x+1} = \dfrac{2}{0}$ Does not exist **D**

25) $f(x+\Delta x) \approx f(x) + f'(x)\Delta x$

$\sqrt[3]{997} \approx \sqrt[3]{1000} + \dfrac{1}{3\cdot 1000^{2/3}}(-3) = \dfrac{999}{100}$ **C**

26) In order for $\lim_{x\to c} f(x)$ to not exist and $\lim_{x\to c}|f(x)| = L$ to exist, we must have

$\lim_{x\to c^+} f(x) = -\lim_{x\to c^-} f(x)$ Let $f(x) = \begin{cases} -1 & x<0 \\ 0 & x=0 \\ 1 & x>0 \end{cases}$

I) $\lim_{x\to c^+} f(x) = 1$ & $\lim_{x\to c^-} f(x) = -1$ both exist-True

II) $\lim_{x\to c^+}|f(x)-1| = 0$ and $\lim_{x\to c^-}|f(x)-1| = 2$ Thus $\lim_{x\to c}|f(x)-L|$ does not exist-False

III) $f(0)$ does exist-False

IV) $\lim_{\Delta x\to 0^+}\dfrac{|f(0+\Delta x)-f(0)|}{\Delta x} = \lim_{\Delta x\to 0^+}\dfrac{1}{\Delta x}$ which does not exist-False

Only I is true **A**

27) $\lim_{x\to\infty}\sqrt{Ax^2+12x} - Bx = \lim_{x\to\infty}\dfrac{Ax^2+12x-B^2x^2}{\sqrt{Ax^2+12x}+Bx}$

For the limit to converge, $A = B^2$.

$\lim_{x\to\infty}\dfrac{12x}{\sqrt{B^2x^2+12x}+Bx} = 3 \Rightarrow \dfrac{12}{\sqrt{B^2}+B} = 3$

$B=2$ and $A=4$ **B**

28) $g'(3) = \dfrac{1}{f'(g(3))}$ $f(x)=3$ when $x=1$

$g(3)=1$ $\dfrac{1}{f'(1)} = \dfrac{1}{3(1)^2+1} = \dfrac{1}{4}$ **B**

29) I) $x\cdot|x| = \begin{cases} -x^2 & x<0 \\ x^2 & x\ge 0 \end{cases}$ – YES

II) $(x-1)\cdot|x| = x\cdot|x| - |x|$; $x\cdot|x|$ works, but $|x|$ is not differentiable at $x=0$ – NO

III) $|x-1|^3 = \begin{cases} (1-x)^3 & x<1 \\ (x-1)^3 & x\ge 1 \end{cases}$ – YES

IV) $x^3\cdot|x| = \begin{cases} -x^4 & x<0 \\ x^4 & x\ge 0 \end{cases}$ –YES **C**

30) $\lim_{x\to 0}\dfrac{x\sin(x)}{\exp(x^2)-1} = \lim_{x\to 0}\dfrac{x\cos(x)+\sin(x)}{2x\exp(x^2)} =$

$\lim_{x\to 0}\dfrac{-x\sin(x)+2\cos(x)}{4x^2\exp(x^2)+2\exp(x^2)} = 1$ **D**