2008 FAMAT State Convention
Interschool Test Solutions

1. The number chosen was 15, guessing 1 as the tens digit and 5 as the ones digit each earns half a point.
2. The best way to do this is through brute force on a computer. The first two are 635318657 and 3262811042. The third is $292^4 + 193^4 = 257^4 + 256^4 = 8657437697$.
3. The n^{th} even term is the n^{th} prime while the n^{th} odd term is $\phi(n)$, so the next 5 terms are 31, 4, 37, 12, and 41.
4. The constant is π. Ptolemy used a 360-gon to approximate π as 3.1416666… Vitruvius approximated π as $\frac{25}{8} = 3.125$. Fibonacci approximated π as 3.141818. Brahmagupta approximated π as $\sqrt{10} = 3.162...$ which was the worst
5. Let $a_i = 1$ if the i^{th} restaurant I try becomes my favorite (meaning it is better than each of the previous ones) $a_i = 0$ otherwise. So the expected number of favorites I have is $E[\sum_{i=1}^{n} a_i] = \sum_{i=1}^{n} E[a_i] = \sum_{i=1}^{n} \Pr[a_i = 1]$ and $\Pr[a_i] = \frac{1}{i}$, so the expected number of favorites I have is $\sum_{i=1}^{n} \frac{1}{i}$ which goes as $\ln(n) + \gamma$ as $n \to \infty$.
6. Both cubes need to have 1 and 2 on them; otherwise 11 and 22 aren't possible. Furthermore, since no one die will have each digit 1 to 9, both cubes need to have 0 to be able to make each of 01 to 09. This leaves 6 sides left, but there are 7 digits remaining (3 through 9). However, since 6 and 9 are never both used at the same time, and a 6 can turn into a 9 by rotating it, it is sufficient to just include one and rotate it depending on whether you need a 6 or a 9. There are many solutions at this point, one of which is: first die: 0, 1, 2, 3, 4, 5; second die: 0, 1, 2, 6, 7, 8. Any of these solutions will only have 5 prime faces, the two 2's, the one 3, the one 5, and the one 7.
7. This can be found by either writing a program or on the internet. The millionth prime is 15485863.
8. This problem can be solved relatively easily by brute forcing it with a computer or finding a table of $\phi(n)$ online.
9. Trivially, $f_0(x) = \sum_{n=1}^{\infty} \frac{1}{x^n} = \frac{\frac{1}{x}}{1-\frac{1}{x}} = \frac{1}{x-1}$. Then,

$$\frac{d}{dx} f_k(x) = \sum_{n=1}^{\infty} \frac{d}{dx} \frac{n^k}{x^n} = -n\sum_{n=1}^{\infty} \frac{n^k}{x^{n+1}} = -\sum_{n=1}^{\infty} \frac{n^{k+1}}{x^{n+1}} = -\frac{1}{x}\sum_{n=1}^{\infty} \frac{n^{k+1}}{x^n} = -\frac{1}{x} f_{k+1}(x) \to$$

$$f_{k+1}(x) = -x \frac{d}{dx} f_k(x)$$

10. The first few numbers spoken are 9, 1**3**, 6, 1, 322, 491, 15**8**102
11. $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \sqrt{i}}{n\sqrt{n}} = \frac{1}{n}\sum_{i=1}^{n} \sqrt{\frac{i}{n}} = \int_0^1 \sqrt{x} = \frac{2}{3}$

12. We should look at this data as just the number of flips that landed on heads and tails rather than individual trials. Obviously, there were 1000 heads, since each trial ended in one. The number of tails is
$0*725+1*199+2*54+3*16+4*4+5*2=381$. So out of 1381 flips, the probability of getting 1000 heads is $\binom{1381}{1000}p^{1000}(1-p)^{381}$, which is maximized when it's derivative equals 0, so
$1000\binom{1381}{1000}p^{999}(1-p)^{381}-381\binom{1381}{1000}p^{1000}(1-p)^{380}=0\rightarrow$
$1000(1-p)=381p\rightarrow p=\dfrac{1000}{1381}\approx .724$
13. Trial and error along with plotting the data and peeling away higher order terms shows that the data fits $f(x)=x^4+2x^3+x+3$, so $f(7)=3097$
14. This is a simple Caesar Shift of 22 spaces with a catch: the space is treated as a 27th letter. The two quotes decode to "God does not care about our mathematical difficulties. He integrates empirically." and "I don't believe in mathematics." which are both commonly attributed to Albert Einstein.
15. The 2002 Mu Alpha Theta National Convention shirts approximated π as 3.1451926…, which is 0.0036 off from the conventional value of 3.1415926…
16. The trick here is to pick an arbitrary point $(x_0,x_0^{\,3})$ on the curve $y=x^3$ and reflect it about the line $y=-3x+1$. Let's say that $(x_0,x_0^{\,3})$ is going to reflect to (x_1,y_1). We know that both points fall on the line $y=\frac{1}{3}(x-x_0)+x_0^{\,3}$, which is the line perpendicular to the line of reflection that goes through the point we are reflecting. Given this and the fact that the distance from $(x_0,x_0^{\,3})$ to $y=-3x+1$ equals half the distance from $(x_0,x_0^{\,3})$ to (x_1,y_1) and lots of messy algebra gives the equation of the curve:
$-256x^3-576x^2y-432xy^2-108y^3+576x^2+864xy+324y^2-132x-724y+8=0$

17. The digits can be mapped to letters using the mapping on the telephone buttons. Each of those combinations of letters can only map to a few different English words, and taking context into consideration, it become clear that the sentence is "Find the sum of ninety and nine" which is 99.
18. Let's call the radius of the polygon (the distance from the center of the polynomial to a vertex, which is also the radius of the circumscribed circle) r. The polygon can be broken down into n identical isosceles triangles with two sides of length r with an angle of $\dfrac{2\pi}{n}$ between them, so the area of each triangle is $\dfrac{r^2\sin(\frac{2\pi}{n})}{2}$, so the area of the polygon is $\dfrac{nr^2\sin(\frac{2\pi}{n})}{2}$. The area of the circumscribed circle is πr^2, so the ratio is $\dfrac{nr^2\sin(\frac{2\pi}{n})}{2\pi r^2}=\dfrac{\sin(2\pi/n)}{2\pi/n}$.

19. Let $y=\lim_{n\to\infty}\prod_{i=1}^{n}\sqrt[n]{\frac{i}{n}}$, $\ln y=\lim_{n\to\infty}\ln(\prod_{i=1}^{n}\sqrt[n]{\frac{i}{n}})=\lim_{n\to\infty}\sum_{i=1}^{n}\ln(\sqrt[n]{\frac{i}{n}})=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln(\frac{i}{n})$ which is the Riemann sum for $\int_0^1\ln(x)=x\ln x-x\Big|_0^1=-1\to y=\frac{1}{e}$

20. Two number are relatively prime if and only if they don't share any prime factors, so for any prime, the probability that two randomly chosen integers share a factor of p is $\frac{1}{p^2}$, so the probability that they don't is $1-\frac{1}{p^2}$. Since we want this to be true for all primes, we want to multiply over them, so the probability of two numbers being relatively prime is $\prod_{primes}(1-\frac{1}{p^2})=\frac{1}{\zeta(2)}=\frac{6}{\pi^2}$.

21. $t_n=\frac{n(n+1)}{2}$, $t_{n-1}=\frac{n(n-1)}{2}$, so $t_n+t_{n-1}=\frac{n^2+n+n^2-n}{2}=n^2=s_n$

22. $6{,}300{,}179{,}600=2^4 5^2 11^2 13*17*19*31$, so all of the odd factors of 6,300,179,600 just the factors of $5^2 11^2 13*17*19*31$ that have a sum of
$$\frac{5^3-1}{5-1}*\frac{11^3-1}{11-1}*\frac{13^2-1}{13-1}*\frac{17^2-1}{17-1}*\frac{19^2-1}{19-1}*\frac{31^2-1}{31-1}=664957440$$

23. $x=\sqrt[4]{2}^{2^{\sqrt[4]{2}^{2^{\cdots}}}}=\sqrt[4]{2}^{2^x}\to\log x=2^x\log\sqrt[4]{2}=\frac{2^x}{4}\log 2=2^{x-2}\log 2$. Graphing $y=\log x$ and $y=2^{x-2}\log 2$ shows that they intersect only once, and inspection shows that that intersection comes at $x=2$

24. The shortest period length is the smallest value of x such that $\frac{10^x}{p}-\frac{1}{p}$ is an integer, so $p\,|\,(10^x-1)$, so $10^x\equiv 1\bmod p$. Since p and 10 are relatively prime, $10^{p-1}\equiv 1\bmod p$, by Fermat's Little Theorem, so the smallest period can be no larger than $p-1$. $p=7$ is an example where the shortest period equals $p-1$, so it is obtainable.

25. The city nicknames and respective real names in the correct order are as follows: The City of Five Flags (Pensacola), The City Beautiful (Orlando), The Big Guava (Tampa), City of Palms (Ft. Myers), Venice of America (Ft. Lauderdale), The Magic City (Miami)

26. A nail is an obscure unit of length measure that is approximately 0.05715 meters and a metric second was the unit of time suggested in the metric time movement that is equal to approximate 0.3456 seconds.
$$c=\frac{299792458\,m}{1\sec}*\frac{1\,nail}{0.05715\,m}*\frac{0.3456\sec}{1\,metric\sec}=1.81\text{ x }10^9$$

27. First, note that the since the circle-square combination shares an axis of symmetry with both the square and the triangle, the centers of mass of the triangle, square and combination will all lie on that line as well. Now, the square has area 16 and

the triangle has area $4\sqrt{3}$. The distance between the square's center of mass and the shared edge of the square and the triangle is 2. The distance between the triangle's center of mass and the shared edge is $\frac{2\sqrt{3}}{3}$, so the distance between the center of masses is $\frac{6+2\sqrt{3}}{3}$. The ratio of the distance from the center of mass of the system to the square's center and the system's center to the triangle's center is $4\sqrt{3}:16$, so the distance from the square is $\frac{6+2\sqrt{3}}{3}*\frac{4\sqrt{3}}{16+4\sqrt{3}}=\frac{2+6\sqrt{3}}{13}$

28. The term described is Euclid's Orchard and the episode is "Brutus"
29. 510510, 570570, 690690, 746130, 870870, 881790, 90321, 930930 each have 7 prime factors, so there are 8.
30. Finding this with a computer is pretty straight forward (although quite slow).
31. Let x be the event that I select 1. Let y be the event that I select from the first distribution and $\overline{y}$ be the event that I select the second distribution. $\Pr[x \mid y]$ is the probability of selecting a number 1 standard deviation from the mean, which is 0.158655. Similarly, $\Pr[x \mid \overline{y}]$ is the probability of selecting a number half a standard deviation above the mean which is .308538. $\Pr[x] = .25 * \Pr[x \mid \overline{y}] + .75 * \Pr[x \mid y] = .196126$ and by Bayes' Rule, $\Pr[y \mid x] = \frac{\Pr[x \cup y]}{\Pr[x]} = \frac{.75 * \Pr[x \mid y]}{.196126} = .607...$
32. The intersection of the two circles can be reduced to two quarter circles of radius $\frac{1}{2}$ contained in a square of side length $\frac{1}{2}$. This means that the intersection is the area which is in both quarter circles, so it is the sum of the areas of the quarter circles minus the area of the square, which is $\frac{\pi-2}{8}$.
33. $C = \frac{5}{9}(F-32)$, so we want to solve $F+10 = \frac{5}{9}(F-32) \to \frac{4}{9}F = \frac{-250}{9} \to F = -62.5$ and $F-10 = \frac{5}{9}(F-32) \to \frac{4}{9}F = -\frac{70}{9} \to F = -17.5$ which correspond to $-52.5°C$ and $-27.5°C$ respectively.
34. There are 216 stitches on a regulation baseball
35. $P(heads > fives) = P(fives = 0) * P(heads > 0) + P(fives = 1) * P(heads > 1) *$
$P(fives = 2) * P(heads > 2) + P(fives = 3) * P(heads > 3) = 0.395$
$P(heads < fives) = P(heads = 0) * P(fives > 0) + P(heads = 1) * P(fives > 1) *$
$P(heads = 2) * P(fives > 2) + P(heads = 3) * P(fives > 3) = 0.365$

36. $f_{i+1} = f_i + f_{i+1}$ so
$f_3 + f_5 + ... + f_i = (f_1 + f_2) + (f_3 + f_4) + ... + (f_{i-2} + f_{i-1}) = f_{i+1} - 1 \rightarrow$
$f_1 + f_3 + f_5 + ... + f_i = 1 + f_{i+1} - 1 = f_{i+1}$

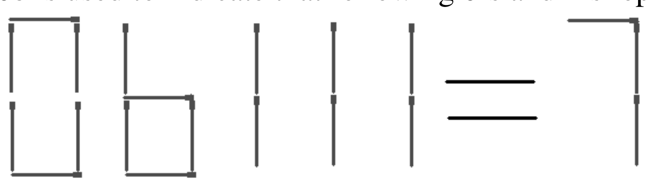
37. $\sum_{i=0}^{\infty}\left(\frac{4i+1}{4i+2} - \frac{4i+3}{4i+4}\right) = -\frac{1}{2}\sum_{i=0}^{\infty}\frac{1}{(2i+1)(2i+2)} = -\frac{\ln 2}{2}$

38. Since the n^{th} triangular number is $\frac{n(n+1)}{2}$, we need one of n or $n+1$ to be a square and the other to be twice a square so their product is a square. There is no other way for $\frac{n(n+1)}{2}$ to be a square, since n and $n+1$ are relatively prime. Looking at a table of primes shows that the 5^{th} smallest square triangular number you can construct is 1413721.

39. By putting restrictions on particular letters: K must by 0 or 1, Y must be even, $E \equiv O \bmod 5$, the following 10 solutions are found: $2169 + 2169 = 04338$, $2943 + 2943 = 05886$, $3279 + 3279 = 06558$, $3614 + 3614 = 07228$, $3941 + 3941 = 07882$, $4831 + 4831 = 09662$, $5386 + 5386 = 10772$, $6385 + 6385 = 12770$, $7832 + 7832 = 15664$, $7943 + 7943 = 15886$.

40. US Patent 5373560 regards cryptography, and the seed in question is the 160-bit seed D1228957 EB28279F 40CD5D9A 901A6A76 FD148B57

41. 0b is used to indicate that following 0's and 1's represent a binary number, so



42. Consider each column to be an element of the sequence, each row, from the bottom up, is a binary bit, where 1's are shaded in and 0's are not. Once seeing this, decoding the first few should give it away. Other clues would be that the sequence is strictly increasing and that the first number is 2 and the rest are odd.