# ***State Convention MA$\Theta$ 2008 – Mu BC Calculus Answers***

**1)** $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\cos t}{3t^2-2t+1}$ $\quad \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$

$=\frac{(-\sin t)(3t^2-2t+1)-(\cos t)(6t-2)}{(3t^2-2t+1)^3}$ Plug in $t=0$:

$\frac{(0)(1)-(1)(-2)}{(1)^3}=2$ **D**

**2)** $\int e^{2x}\sin(e^x)dx=\int(e^x)(e^x\sin(e^x)dx)=-e^x\cos(e^x)-$
$\int(-\cos(e^x))e^x dx=-e^x\cos(e^x)+\sin(e^x)+C$ **A**

**3)** Apply the ratio test: $a_n=\frac{2^n}{n^2x^n}$; $a_{n+1}=\frac{2^{n+1}}{(n+1)^2x^{n+1}}$

For convergence $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$. $\lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)^2x^{n+1}}\cdot\frac{n^2x^n}{2^n}\right|=$

$\lim_{n\to\infty}\left|\frac{2}{x}\cdot\frac{n^2}{(n+1)^2}\right|=\left|\frac{2}{x}\right|<1\Rightarrow|x|>2$ must check endpoints:

$x=2$: $\sum_{n=1}^{\infty}\frac{2^n}{n^2 2^n}=\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges. $x=-2$:

$\sum_{n=1}^{\infty}\frac{2^n}{n^2(-2)^n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ converges. $(-\infty,-2]\cup[2,\infty)$ **C**

**4)** Centroid $=([1+7+4]/3,[1+1+4]/3)=(4,2)\Rightarrow r=2$
$A=(3)(6)/2=9$; $V=2\pi\cdot r\cdot A=36\pi$ **B**

**5)** $\frac{1}{2-1}\int_1^2(\ln(x))(1dx)=x\ln(x)\big|_1^2-\int_1^2(x)\left(\frac{1}{x}dx\right)=$
$2\ln(2)-1\ln(1)-1=2\ln(2)-1$ **D**

**6)** $\sin(x^2)\approx x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}$ $\quad e^{-x^3}\approx 1-x^3+\frac{x^6}{2!}-\frac{x^9}{3!}$

$\lim_{x\to0}\frac{x^2-\left(x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}\right)}{1-x^3-\left(1-x^3+\frac{x^6}{2!}-\frac{x^9}{3!}\right)}=$

$\lim_{x\to0}\frac{\frac{x^6}{3!}-\frac{x^{10}}{5!}}{-\frac{x^6}{2!}+\frac{x^9}{3!}}=\lim_{x\to0}\frac{\frac{1}{3!}-\frac{x^4}{5!}}{-\frac{1}{2!}+\frac{x^3}{3!}}=-\frac{1}{3}$ **B**

**7)** $\int_0^1\frac{1}{\sqrt{1-x^2}}dx$ is an improper integral, thus
$=\lim_{a\to0}\left(\arcsin(x)\big|_a^1\right)=\pi/2-\lim_{a\to0}\arcsin(a)=\pi/2$ **E**

**8)** There are $k$ petals if $k$ is odd and $2k$ petals is $k$ is even. Thus, there can never be a 6 petaled rose. **C**

**9)** I. $3^n-2^n<3^n\Rightarrow\frac{1}{3^n-2^n}>\frac{1}{3^n}$ for all $n$. The direct comparison test cannot be applied.

II. $\lim_{n\to\infty}\frac{3^n-2^n}{3^n}=\lim_{n\to\infty}1-\left(\frac{2}{3}\right)^n=1<\infty$ Since $\sum_{n=1}^{\infty}\frac{1}{3^n}=\frac{1}{2}$ converges, then $\sum_{n=1}^{\infty}\frac{1}{3^n-2^n}$ converges.

III. $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{3^n-2^n}{3^{n+1}-2^{n+1}}=$

$\lim_{n\to\infty}\frac{1-(2/3)^n}{3-2(2/3)^n}=\frac{1}{3}<1\Rightarrow\sum_{n=1}^{\infty}\frac{1}{3^n-2^n}$ converges.

IV. $\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\frac{1}{3^n-2^n}}=$

$\lim_{n\to\infty}\frac{1}{3}\frac{1}{\sqrt[n]{1-(2/3)^n}}=\frac{1}{3}<1\Rightarrow\sum_{n=1}^{\infty}\frac{1}{3^n-2^n}$ converges. **A**

**10)** $f(x)=\sin(x^2+\pi/6)-x$, $f'(x)=2x\cos(x^2+\pi/6)-1$,
$f(0)=1/2$, $f'(0)=-1$ $P_1(x)=f(0)+f'(0)x=1/2-x$ **A**

**11)** $\frac{e^x}{1+e^{-x}}=\frac{1+e^x-1}{1+e^{-x}}=\frac{1+e^x}{1+e^{-x}}-\frac{1}{1+e^{-x}}=$
$\frac{e^x(1+e^x)}{e^x(1+e^{-x})}-\frac{e^x}{e^x(1+e^{-x})}=\frac{e^x(1+e^x)}{1+e^x}-\frac{1}{1+e^x}$

$\int\frac{e^x dx}{1+e^x}=\int e^x dx-\int\frac{dx}{1+e^x}=e^x-\ln(1+e^x)+C$ **B**

**12)** $s=\int_a^b\sqrt{(dx/dt)^2+(dy/dt)^2}\,dt=\int_0^{\pi}\sqrt{(-\sin t)^2+(\cos t)^2}\,dt$

$\int_0^{\pi}dt=\pi$ **C**

**13)** $\sum_{n=1}^{\infty}\left[\frac{1}{n}\sin\left(\frac{\pi}{2}n\right)\right]=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$ By the alternating series test $\sum_{n=1}^{\infty}\left[\frac{1}{n}\sin\left(\frac{\pi}{2}n\right)\right]$ converges $\sum_{n=1}^{\infty}\left|\frac{1}{n}\sin\left(\frac{\pi}{2}n\right)\right|=$

$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{2n-1}\right|=\sum_{n=1}^{\infty}\frac{1}{2n-1}$ diverges. $\sum_{n=1}^{\infty}\left[\frac{1}{n}\sin\left(\frac{\pi}{2}n\right)\right]$ conditionally converges. **B**

**14)** $g(x_{n+1})=g(x_n)+\Delta x\cdot g'(x_n)$
$g(0.25)=0+(0.25)(-1)=-0.25$,
$g(0.5)=-0.25+(0.25)(-3)=-1$
$g(0.75)=-1+(0.25)(4)=0$, $g(1)=0+(0.25)(2)=0.5$ **C**

**15)** $\frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+f(x_3)\right]=$

$\frac{4-1}{2(3)}\left[f(1)+2f(2)+2f(3)+f(4)\right]=$

$\frac{1}{2}\left[1+2(3)+2(9)+63\right]=44$ **E**

**16)** $\frac{1}{(x+a)(x+2a)(x+3a)}=\frac{A}{x+a}+\frac{B}{x+2a}+\frac{C}{x+3a}$

$A=C=\frac{1}{2a^2}\ \ B=-\frac{1}{a^2}$,

$\frac{1}{2a^2}\int\frac{dx}{x+a}-\frac{1}{a^2}\int\frac{dx}{x+2a}+\frac{1}{2a^2}\int\frac{dx}{x+3a}$

$=\frac{1}{2a^2}\left[\ln|x+a|-2\ln|x+2a|+\ln|x+3a|\right]+C$ **D**

**17)** $f'(x)=-1/\sqrt{1-x^2}$ $\qquad f''(x)=-x/\left(1-x^2\right)^{3/2}$

I. TRUE $f''(x)<0$ for $-1\le x\le 0$; II. FALSE $f'(x)$ does not have local minima on the interval; III. TRUE $f'(x)\ne 0$ for $-1\le x\le 1$; IV. TRUE $f''(0)=0$ which is a max. **A**

**18)** $\lim_{x\to 0}\frac{2^x\sin(x)}{\left(2^x-1\right)\cos(x)}=\lim_{x\to 0}\frac{2^x\tan(x)}{2^x-1}=$

$\lim_{x\to 0}\frac{2^x\sec^2(x)+\ln(2)2^x\tan(x)}{\ln(2)2^x}=\frac{1}{\ln(2)}$ **A**

**19)** $r(\theta)=3+3\cos(\theta)$ is a cardioid, so we only need to integrate from 0 to $\pi$, then double the value.

$A=(2)\frac{1}{2}\int_0^\pi[3+3\cos(\theta)]^2\,d\theta=\int_0^\pi\left(9+18\cos(\theta)+9\cos^2(\theta)\right)d\theta$

$9\pi+18\sin(\theta)\Big|_0^\pi+\frac{9}{2}\int_0^\pi(1+\cos(2\theta))d\theta=$

$9\pi+0+\frac{9}{2}\left(\theta+\frac{1}{2}\sin(2\theta)\right)\Big|_0^\pi=9\pi+\frac{9\pi}{2}=\frac{27\pi}{2}$ **C**

**20)** Let $u=\sqrt{x}$: $du=1/\left(2\sqrt{x}\right)dx$, $dx=2udu$

$\int_0^\infty\left(\sqrt{x}\right)e^{-\sqrt{x}}dx=\int_0^\infty ue^{-u}(2udu)=2\int_0^\infty u^2e^{-u}du=$

$2\left[-u^2e^{-u}-2ue^{-u}-2e^{-u}\right]_0^\infty=4$ **D**

**21)** $\frac{1}{y^2+1}\frac{dy}{dx}=\frac{4}{3}\frac{1}{x^2}\Rightarrow\tan^{-1}(y)=-\frac{4}{3x}+C\Rightarrow$

$y=\tan\left(-\frac{4}{3x}+C\right)=-\tan\left(\frac{4}{3x}+C\right)$ **B**

**22)** $f(t)=\frac{1}{1-t}=\sum_{n=0}^\infty t^n$ ; $f'(t)=\frac{1}{(1-t)^2}=\sum_{n=0}^\infty nt^{n-1}$

$tf'(t)=\frac{t}{(1-t)^2}=\sum_{n=0}^\infty nt^n$ ; $x^3f'\left(x^3\right)=\frac{x^3}{\left(1-x^3\right)^2}=\sum_{n=0}^\infty nx^{3n}$ **C**

**23)** $\operatorname{arccsc}(x)=\arcsin(1/x)$; $\frac{d}{dx}(\arcsin(1/x))=\frac{-1/x^2}{\sqrt{1-(1/x)^2}}$,

at $x=\sqrt{2}$: $\frac{-1/2}{\sqrt{1-1/2}}=-\frac{\sqrt{2}}{2}$ **B**

**24)** $g'(x)=x\cosh(x)$ $\qquad g'(x)=0$ only when $x=0$.
$g''(x)=x\sinh(x)+\cosh(x)>0$ for all $x$ which means $g(x)$ is concave up for all $x$. $\quad g''(0)=1\Rightarrow x=0$ is a minimum. $g(0)=-1$. Range is $[-1,\infty)$ **A**

**25)** 1) $a_n=1/n$. $\sum_{n=1}^\infty\frac{1}{n}$ diverges, but $\sum_{n=1}^\infty\frac{1}{n^2}$ converges. 2) $a_n=n$. $\sum_{n=1}^\infty n$ diverges and $\sum_{n=1}^\infty 1$ diverges. **D**

**26)** $\frac{dy}{dx}=\frac{r'(\theta)\sin\theta+r(\theta)\cos\theta}{r'(\theta)\cos\theta-r(\theta)\sin\theta}$

$r(\theta)=1-\cos(2\theta)$
$r'(\theta)=2\sin(2\theta)$ $\quad\frac{dy}{dx}=\frac{\left(\sqrt{3}\right)\left(\sqrt{3}/2\right)+(3/2)(1/2)}{\left(\sqrt{3}\right)(1/2)-(3/2)\left(\sqrt{3}/2\right)}=-3\sqrt{3}$ **A**

**27)** $\int_0^x\exp\left(-2t^2\right)dt\approx\int_0^x\left[1-2t^2+\left(2t^2\right)^2/2!\right]dt=$

$\int_0^x\left[1-2t^2+2t^4\right]dt=x-\frac{2}{3}x^3+\frac{2}{5}x^5$ **B**

**28)** $\frac{d}{dx}\left(\frac{1}{x^k}\right)=-\frac{k}{x^{k+1}}\Rightarrow-x\left[\frac{d}{dx}\left(\frac{1}{x^k}\right)\right]=\frac{k}{x^k}$

$\sum_{k=0}^\infty-x\left[\frac{d}{dx}\left(\frac{1}{x^k}\right)\right]=-x\left[\frac{d}{dx}\left(\sum_{k=0}^\infty\left(\frac{1}{x}\right)^k\right)\right]=-x\left[\frac{d}{dx}\left(\frac{1}{1-1/x}\right)\right]=$

$=\frac{x}{(x-1)^2}\qquad\frac{3}{(3-1)^2}=\frac{3}{4}$ **C**

**29)** $\bar{x}=\frac{\int xf(x)dx}{\int f(x)dx}\qquad\int_1^e\ln(x)dx=x\ln x-x\Big|_1^e=1$

$\int_1^e x\ln(x)dx=\frac{x^2}{4}\ln x-\frac{x^2}{4}\Bigg|_1^e=\frac{e^2+1}{4}$ **D**

**30)** $a_1=2$, $a_2=3/2$, $a_3=4/3$, $a_4=5/4$ The sequence is bounded above by 2 and bounded below by 1. Since $a_1>a_2>a_3>\ldots>a_n>\ldots$, the sequence is monotonically decreasing. **A**