CALCULUS INDIVIDUAL SOLUTIONS

1. $f'(x)=6x$. $f'(4)=24$. $f(4)=8$. $y=24(x-4)+8=24x-88$. **A**

2. $\lim_{x\to 3}\sqrt{\frac{x^2-5x+6}{x^2-2x-3}}=\lim_{x\to 3}\sqrt{\frac{(x-2)(x-3)}{(x-3)(x+1)}}=\lim_{x\to 3}\sqrt{\frac{x-2}{x+1}}=\sqrt{\frac{1}{4}}=\frac{1}{2}$. **E**

3. $y^2-4y+5=y+1$. $y^2-5y+4=0$. $y=1$ and $y=4$.

$$\int_1^4\left(y+1-y^2+4y-5\right)dx=\int_1^4\left(-y^2+5y-4\right)dx=\left[-\frac{y^3}{3}+\frac{5y^2}{2}-4y\right]_1^4=-21+\frac{75}{2}-12=\frac{9}{2}.$$ **D**

4. $f'(x)\ge 0$, $f''(x)\ge 0$, $g'(x)\le 0$, $g''(x)\le 0$ for all x. $\frac{d}{dx}f(g(x))=f'(g(x))g'(x)\le 0$.

$\frac{d^2}{dx^2}f(g(x))=f''(g(x))\left[g'(x)\right]^2+f'(g(x))g''(x)\ge 0$. **C**

5. $\int_0^1\frac{3x^3+5x^2+x}{x+1}dx=\int_0^1\left(3x^2+2x-1+\frac{1}{x+1}\right)dx=\left[x^3+x^2-x+\ln(x+1)\right]_0^1=1+\ln 2$. **A**

6. $$\lim_{x\to\infty}\frac{\left(ax^4+bx^3+cx^2+dx+e\right)-\left(x^4+3x^3-2x^2-4x-7\right)}{\sqrt{ax^4+bx^3+cx^2+dx+e}+\sqrt{x^4+3x^3-2x^2-4x-7}}=$$

$$\lim_{x\to\infty}\frac{\left(ax^2+bx+c+\frac{d}{x}+\frac{e}{x^2}\right)-\left(x^2+3x-2-\frac{4}{x}-\frac{7}{x^2}\right)}{\sqrt{a+\frac{b}{x}+\frac{c}{x^2}+\frac{d}{x^3}+\frac{e}{x^4}}+\sqrt{1+\frac{3}{x}-\frac{2}{x^2}-\frac{4}{x^3}-\frac{7}{x^4}}}=c.$$ If the limit converges, it follows that $a=1$ and $b=3$. $$\lim_{x\to\infty}\frac{\left(c+\frac{d}{x}+\frac{e}{x^2}\right)-\left(-2-\frac{4}{x}-\frac{7}{x^2}\right)}{\sqrt{1+\frac{2}{x}+\frac{c}{x^2}+\frac{d}{x^3}+\frac{e}{x^4}}+\sqrt{1+\frac{3}{x}-\frac{2}{x^2}-\frac{4}{x^3}-\frac{7}{x^4}}}=\frac{c+2}{2}=c.$$ $c=2$. **B**

7. $y=\frac{x}{1+y}$. $y^2+y-x=0$. $y=\frac{-1+\sqrt{1+4x}}{2}$. $y'=\frac{1}{\sqrt{1+4x}}$. $y'(6)=\frac{1}{5}$. **C**

8. $$\int_2^6\frac{x}{1+\frac{x}{1+\dots}}dx=\int_2^6\frac{-1+\sqrt{1+4x}}{2}dx=\left[-\frac{x}{2}+\frac{1}{12}(1+4x)^{\frac{3}{2}}\right]_2^6=-2+\frac{98}{12}=\frac{37}{6}.$$ **D**

9. Let the area of the rectangle after t seconds be given by $A(t)$. Then $A(t)=(5-t)(2+2t)=-2t^2+8t+10$. $A'(t)=-4t+8=0$. $t=2$. Since $A''(2)=-2<0$, $A(t)$ attains a local maximum $A(2)=-2\cdot 4+8\cdot 2+10=18$ when $t=2$. $A(0)=10$ and $A(5)=0$, so the maximum area of the rectangle is 18 square inches. **C**

10. If either $f(0)=0$ or $f(1)=1$, then the value of *c* is obvious. Let $g(x)=f(x)-x$. If $f(0)\neq 0$, then $g(0)>0$. If $f(1)\neq 1$, then $g(1)<0$. By the Intermediate Value Theorem, there exists a number *c* in $[0,1]$ such that $g(c)=0$, and hence $f(c)=c$. **A**

11. $f(x)=x^{\frac{1}{3}}$. $f'(x)=\frac{1}{3}x^{-\frac{2}{3}}$. $f(500)\approx f(512)-12\cdot f'(512)=8-12\cdot\frac{1}{3}\cdot\frac{1}{64}=7.9375$. **C**

12. $\frac{1}{\frac{8}{3}-\frac{2}{3}}\int_{\frac{2}{3}}^{\frac{8}{3}}\frac{1}{x}\,dx=\frac{1}{2}\left[\ln x\right]_{\frac{2}{3}}^{\frac{8}{3}}=\frac{1}{2}\ln 4=\ln 2$. **B**

13. $\sum_{i=0}^{\infty}\frac{x^i}{i!}=e^x$, so $\sum_{i=0}^{\infty}\frac{(\ln 2)^i}{i!}=e^{\ln 2}=2$. **C**

14. $f(x)$ is continuous at $x=1$ if $1^2+a\cdot 1+b=-1^2+8\cdot 1+4$ and hence $a+b=10$. $f(x)$ is differentiable at $x=1$ if $2\cdot 1+a=-2\cdot 1+8$ and hence $a=4$. $b=6$. $a\cdot b=24$. **C**

15. $\int_0^2 f(x)\,dx=\int_0^1\left(x^2+ax+b\right)dx+\int_1^2\left(-x^2+8x+4\right)dx=\left[\frac{x^3}{3}+\frac{ax^2}{2}+bx\right]_0^1+\left[-\frac{x^3}{3}+4x^2+4x\right]_1^2$

$=\frac{1}{3}+\frac{a}{2}+b-\frac{7}{3}+12+4=\frac{a}{2}+b+14=20$. $\frac{a}{2}+b=6$. $a+2b=12$. **B**

16. $f'(x)=2xe^{x^2}+1$. $f''(x)=2e^{x^2}+4x^2e^{x^2}$. $f(0)=1$. $f'(0)=1$. $f''(0)=2$.

$\int_0^{1/2}\left(e^{x^2}+x\right)dx\approx\int_0^{1/2}\left(1+x+x^2\right)dx=\left[x+\frac{x^2}{2}+\frac{x^3}{3}\right]_0^{\frac{1}{2}}=\frac{1}{2}+\frac{1}{8}+\frac{1}{24}=\frac{2}{3}$. **B**

17. $f'(x)=\dfrac{\left(2x\sin(x)+x^2\cos(x)+3\right)\left(1+\ln\left(x^2+1\right)\right)-\left(x^2\sin(x)+3x\right)\frac{2x}{x^2+1}}{\left[1+\ln\left(x^2+1\right)\right]^2}$.

$f'(0)=\frac{3\cdot 1-0\cdot 0}{[1+0]^2}=3$. **C**

18. $\sum_{i=1}^{\infty}\frac{|a_i|}{N}=1$, so $\frac{|a_i|}{N}\leq 1$ for all *i*. Then $\left(\frac{|a_i|}{N}\right)^2\leq\frac{|a_i|}{N}\leq 1$, so $\sum_{i=1}^{\infty}\frac{(a_i)^2}{N^2}\leq 1$ and hence $\sum_{i=1}^{\infty}(a_i)^2\leq N^2$. Then $\sum_{i=1}^{\infty}(a_i)^2$ converges. Consider $\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i=1$. However, $\sum_{i=1}^{\infty}\left(\frac{1}{4}\right)^i=\frac{1}{3}$, so $\sum_{i=1}^{\infty}(a_i)^2$ does not have to converge to an integer. **B**

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19. $\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}$. $\int_{-\infty}^{\infty} e^{-x^2+4x-3}dx=\int_{-\infty}^{\infty} e^{-(x-2)^2+1}dx=e\int_{-\infty}^{\infty} e^{-x^2}dx=e\sqrt{\pi}$. **B**

20. Taking derivatives on both sides, $f\left(x^2\right)+2x^2f'\left(x^2\right)-f\left(4\right)=f\left(2x\right)+2xf'\left(2x\right)-4f'\left(2x\right)$.

Evaluating when $x=0$, $f\left(0\right)-f\left(4\right)=f\left(0\right)-4f'\left(0\right)$. $f'\left(0\right)=\frac{f\left(4\right)}{4}=\frac{12}{4}=3$. **C**

21. $\ln 2=\frac{1}{x}\ln 2^x$. $\frac{d}{dx}\left[\frac{2^x}{x}\right]=\frac{x\ln 2\cdot 2^x-2^x}{x^2}=0$. $x=\frac{1}{\ln 2}$. $\ln 2\cdot 2^{\frac{1}{\ln 2}}=e\ln 2$. **B**

22. $\frac{d}{dx}\int_x^{2x}\ln t^3\,dt=2\ln\left(8x^3\right)-\ln x^3=\ln\left(64x^6\right)-\ln x^3=\ln\left(64x^3\right)=3\ln\left(4x\right)$. **D**

23. $\frac{dy}{dx}=-\left[x^{\alpha-1}+1\right]^{-2}\cdot\left(\alpha-1\right)x^{\alpha-2}>0$ when $x=2$. $\frac{d^2y}{dx^2}=2\left[x^{\alpha-1}+1\right]^{-3}\cdot\left(\alpha-1\right)^2x^{2\alpha-4}$
$-\left[x^{\alpha-1}+1\right]^{-2}\cdot\left(\alpha-1\right)\left(\alpha-2\right)x^{\alpha-3}=\left(\alpha-1\right)x^{\alpha-3}\left[x^{\alpha-1}+1\right]^{-3}\left[2\left(\alpha-1\right)x^{\alpha-1}-\left(\alpha-2\right)\left(x^{\alpha-1}+1\right)\right]$
$=\left(\alpha-1\right)x^{\alpha-3}\left[x^{\alpha-1}+1\right]^{-3}\left[\alpha x^{\alpha-1}+2-\alpha\right]<0$ when $x=2$. **B**

24. Let $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{2x}=y$. Then $\lim_{x\to\infty}2x\ln\left(1+\frac{3}{x}\right)=\ln y$. $\lim_{x\to\infty}\frac{2\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}}=\ln y$. $\lim_{x\to\infty}\frac{2\frac{-\frac{3}{x^2}}{1+\frac{3}{x}}}{-\frac{1}{x^2}}=\ln y$.

$\lim_{x\to\infty}\frac{6}{1+\frac{3}{x}}=\ln y$. $\ln y=6$. $y=e^6$. **D**

25. $y=10-x$. We want to minimize $x^2-x\left(10-x\right)+2\left(10-x\right)=2x^2-12x+20$. $4x-12=0$.
$x=3$. $2\cdot 3^2-12\cdot 3+20=18-36+20=2$. **A**

26. $f\left(x\right)=\left[\frac{\ln x}{\ln 2}\right]^2$. $f'\left(x\right)=\frac{2}{\left[\ln 2\right]^2}\frac{\ln x}{x}$. $f'\left(2\right)=\frac{2}{\left[\ln 2\right]^2}\cdot\frac{\ln 2}{2}=\frac{1}{\ln 2}=\log_2 e$. **B**

27. $\lim_{n\to\infty}\sum_{i=0}^{n}\frac{n}{n^2-2in+2i^2}=\lim_{n\to\infty}\sum_{i=0}^{n}\frac{1}{1-2\frac{i}{n}+2\left(\frac{i}{n}\right)^2}\frac{1}{n}=\int_0^1\frac{1}{2x^2-2x+1}dx=\int_0^1\frac{1}{2\left(x-\frac{1}{2}\right)^2+\frac{1}{2}}dx$

$=\int_0^1\frac{1}{2\left(x-\frac{1}{2}\right)^2+\frac{1}{2}}dx=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{2}{4x^2+1}dx=\left[\arctan\left(2x\right)\right]_{-\frac{1}{2}}^{\frac{1}{2}}=\frac{\pi}{2}$. **D**

28. Let x be the number of widgets in excess of 50. Then revenue is $(50+x)(100-x)=5000+50x-x^2$. $-2x+50=0$. $x=25$. $x+50=75$. **C**

29. It can be shown that $f^{(n)}(x)=\left(x^2+2nx+n(n-1)\right)e^x$. $f^{(10)}(10)=(100+200+90)e^{10}=390e^{10}$. **D**

30. $\lim\limits_{x\to 4}\frac{x^2+4x}{x+4}=\frac{4^2+4\cdot 4}{4+4}=\frac{32}{8}=4$. **E**