Integration Topic Test

FAMAT State Convention 2000

For all questions, answer E. "NOTA" means "none of these answers."
[ ] is a grouping symbol only, not the greatest integer function.
"DNE" stands for "does not exist."

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1. Find the mean value of $f(x) = 2^x$ on the interval [1, 2000].

   A. $\frac{2^{2000}-2}{1999}$ B. $\frac{2^{2000}-2}{1999 \ln 2}$ C. $\frac{2^{1999}-1}{1000 \ln 2}$ D. $\frac{2^{2000}-2}{\ln 2}$ E. NOTA

2. Find the area bounded by the graphs of $y = x^3$ and $y = \sqrt[3]{x}$.

   A. 1 B. 3/4 C. 1/2 D. 2/3 E. NOTA

3. Evaluate:

$$\int \tan x \sin^2 x \cos^3 x \, dx$$

   A. $\frac{1}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$
   B. $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$
   C. $\frac{1}{5}\sin^5 x - \frac{1}{3}\sin^3 x + C$
   D. $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$
   E. NOTA

4. Let $A = \int_1^\infty x^{-2} dx$, $B = \int_0^\infty e^{-2x} dx$. Find $A - B$.

   A. -3/2 B. 1/2 C. 1/6 D. -3/2 E. NOTA

5. Let $F(x) = \int_x^{x^3} t^3 \ln t dt$. Evaluate $F'(x)$.

   A. $x^3(3x^6 - 1)\ln x$ B. $3x^3(x^6 - 1)\ln x$ C. $x^3(9x^8 - 1)\ln x$ D. $x^9 \ln x^3$ E. NOTA

6. Evaluate:

$$\int e^x \sqrt{1 - e^x} dx$$

   A. $\frac{2}{3}(1 - e^x)^{3/2} + C$
   B. $-\frac{2}{3}(1 - e^x)^{3/2} + C$
   C. $2(1 - e^x)^{-1/2} + C$
   D. $-2(1 - e^x)^{-1/2} + C$
   E. NOTA

7. Evaluate: $\int_{-2}^{7} \frac{dx}{x}$

A. $\ln \frac{7}{2}$ B. $2\ln 3$ C. $\ln 14$ D. DNE E. NOTA

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous solution of the functional equation $f(1-x) = f(x) + (1-3x)$. Evaluate: $\int_{0}^{1} f(x)dx$

A. No such f exists B. 1/4 C. -1/4 D. 1/2 E. NOTA

9. Evaluate: $\int \tan^{-1} x \, dx$

A. $\frac{1}{2}(\tan^{-1} x)^2 + C$
B. $x\tan^{-1} x - x\ln(x^2+1) + C$
C. $x\tan^{-1} x - \frac{1}{2}\ln(x^2+1) + C$
D. $x\tan^{-1} x - \ln(x^2+1) + C$
E. NOTA

10. Let $f(x) = (x-2)^2 + 1$, and let R be the region between the graph of f on [2,3] and the x-axis. Which of the following is a correct set-up for evaluating the volume obtained by revolving R about the line $x = -1$?

A. $\pi \int_{2}^{3} [(x-2)^2+1]^2 - (-1)^2 dx$ B. $\pi \int_{2}^{3} [((x-2)^2+1) - (-1)]^2 dx$
C. $2\pi \int_{2}^{3} x((x-2)^2+1)dx$ D. $2\pi \int_{2}^{3} (x+1)((x-2)^2+1)dx$
E. NOTA

11. Evaluate: $\int_{0}^{2\pi} \sqrt{1-\cos 2x}\,dx$

A. $2\sqrt{2}$ B. $8\sqrt{2}$ C. 0 D. 4 E. NOTA

12. A ball is thrown down from the top of a 144 foot tall building with initial velocity -28 ft/sec. Assuming the rate of gravitational acceleration to be -32 ft/sec^2, find the mean height above the base of the building of the ball during its fall to the ground.

A. 96 ft B. 85.5 ft C. 72 ft D. 79.5 ft E. NOTA

13. Evaluate: $\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{n}(\frac{2i}{n}+3)^3$

A. 68　　B. $\frac{137}{2}$　　C. 125　　D. $\frac{113}{2}$　　E. NOTA

14. Evaluate: $\int e^{2x}\cos(3x)dx$

A. $\frac{1}{3}e^{2x}\sin(3x)+C$　　B. $\frac{1}{5}e^{2x}\cos(3x)+\frac{2}{5}e^{2x}\sin(3x)+C$
C. $\frac{3}{13}e^{2x}\cos(3x)+\frac{2}{13}e^{2x}\sin(3x)+C$　　D. $\frac{2}{13}e^{2x}\cos(3x)+\frac{3}{13}e^{2x}\sin(3x)+C$
E. NOTA

15. Approximate $\int_1^{10}\sin(x^2)dx$ by Simpson's rule with $n=10$, *rounded* to 3 decimal places.

A. 1.162　　B. 0.273　　C. -1.298　　D. 1.230　　E. NOTA

16. Find the arclength of the region in the first quadrant given by the graph of the equation $x^{2/3}+y^{2/3}=9$, an "astroid."

A. 27/2　　B. 18　　C. 81/2　　D. DNE　　E. NOTA

17. Find the area of the surface obtained by revolving the first quadrant region from problem 16 about the y-axis.

A. $\frac{104976}{35}\pi$　　B. $\frac{16}{105}\pi$　　C. 81π　　D. $\frac{4374}{5}\pi$　　E. NOTA

18. Find the area of one leaf of the "rose" described by the polar graph $r=\cos 2\theta$.

A. $\pi/4$　　B. $\pi/8$　　C. 1　　D. $3\pi/16$　　E. NOTA

19. Valentine has just finished cooking a pot of soup to be served the next day. The soup is currently at 100° C. Unfortunately, her refrigerator is not powerful enough to accommodate soup warmer than 20° C. If by cooling the pot in a sink full of ice water (remaining essentially constant at 5° C), she can bring the temperature down to 50° C in 10 minutes, how many more minutes will it take for the soup to cool to 20°? Round your answer to the nearest minute.

A. 25 min　　B. 6 min　　C. 15 min　　D. 24 min　　E. NOTA

20. Evaluate $\int_2^7 f(-x)dx$ given the following:

$$\int_3^4 f(x)dx = 2 \quad \int_4^7 f(x)dx = 5 \quad \int_{-4}^{-8} f(x)dx = 3$$

$$\int_3^2 f(x)dx = -2 \quad \int_{-8}^{-7} f(x)dx = -6 \quad \int_{-2}^{-4} f(x)dx = 8$$

A. 17 B. -5 C. 9 D. 5 E. NOTA

21. Let

$$M = \int_0^1 \frac{64dx}{8\sqrt{1-x^2}+e^x} \quad A = \int_0^1 \frac{64x^2+e^{2x}}{8\sqrt{1-x^2}+e^x}dx \quad \Theta = \int_1^{13} \frac{dx}{x}$$

Evaluate $M - A - \Theta$

A. 3/2 B. 2 C. $-\ln 13 - e + 2\pi$ D. $-\ln 13 - e + 2\pi + 1$ E. NOTA

22. Evaluate:

$$\int_0^{\pi/4} \frac{1}{1+\tan x}dx$$

A. 13/23 B. $\frac{\ln 2+\pi}{4}$ C. $\frac{9}{50}\pi$ D. $\frac{2\ln 2+\pi}{8}$ E. NOTA

23. Consider the parallelogram $MATH$ lying in the Cartesian plane with vertices $M = (4,5)$, $A = (3,3)$, $T = (8,4)$, and $H = (9,6)$. Find the volume of the solid obtained by revolving parallelogram $MATH$ about the line $y = -2x - 1$.

A. $70\pi\sqrt{5}$ B. $63\pi\sqrt{5}$ C. $90\pi\sqrt{3}$ D. $140\pi\sqrt{3}$ E. NOTA

24. Which of the following conditions are sufficient for a function $f : [a,b] \to [a,b]$ to be Riemann integrable on $I = [a,b]$?

i) f is monotone on I
ii) f is continuous on I
iii) f is bounded on I
iv) $f = g^2$, where g is a Riemann integrable function on I

A. i, ii, iii B. i, iii, iv C. i, ii, iv D. iv E. NOTA

25. Let f, g be Riemann integrable functions mapping [0,1] into [0,1]. How many of the following functions are necessarily Riemann integrable on [0,1]? $f/g, fg, f+g, f \circ g$ (i.e., the composition of f and g)

A. 1 B. 2 C. 3 D. 4 E. NOTA

26. Peter is practicing throwing darts at a square board (and hits it with probability 1). Suppose Peter focuses his attention on one particular edge/side of the board, say A. What is the probability that on a given throw, Peter's dart lands closer to A than to the center of the board? Assume every point of the board has an equal chance of being hit.

A. 1/3 B. 2/3 C. 1/2 D. 5/16 E. NOTA

27. For positive integers k, define $L_k = \lim_{n\to\infty} \frac{1}{(2n+1)^k} + \frac{1}{(2n+2)^k} + \cdots + \frac{1}{(5n)^k}$. Which of the following is closest to $\sum_{k=1}^{\infty} L_k$?

A. 0 B. 1 C. 10 D. 100 E. NOTA

28. Consider a solid with its base the following ellipse: $4x^2 + 9y^2 = 36$. The cross sections perpendicular to the x-axis are regular hexagons with one side in the xy-plane with its endpoints on the ellipse. Find the volume of the solid.

A. $96\sqrt{3}$ B. 16π C. $54\sqrt{3}$ D. 24π E. NOTA

29. Given that the solid in problem 28 has constant density and is entirely contained in the half-space $z \geq 0$, find its center of mass.

A. $(0,0,2)$ B. $(0,0,\frac{\pi}{2})$ C. $(0,0,\frac{\sqrt{3}}{2})$ D. $(0,0,\frac{\pi\sqrt{3}}{2})$ E. NOTA

30. Find a closed form expression for the following integral: $\int \frac{dx}{x(x-1)\ldots(x-m)}$ where m is a positive integer.

A. $\frac{1}{m!}\ln(\prod_{i=0}^{i=m}|x-i|)+C$

B. $\frac{(-1)^{m-i}}{m!}\ln(\prod_{i=0}^{i=m}|x-i|)+C$

C. $\sum_{i=0}^{i=m}\frac{1}{m!}\binom{m}{i}\ln|x-i|+C$

D. $\sum_{i=0}^{i=m}\frac{(-1)^{m-i}}{m!}\binom{m}{i}\ln|x-i|+C$

E. NOTA