Florida Association of Mu Alpha Theta

INTEGRATION TOPIC TEST SOLUTIONS

2000 State Convention

1. By definition, the mean value is

$$\frac{\int_1^{2000} 2^x dx}{2000-1} = \frac{\int_1^{2000} e^{x\ln 2}dx}{1999} = \frac{\frac{1}{\ln 2}2^{2000} - \frac{1}{\ln 2}2^1}{1999} = \frac{2^{2000}-2}{1999\ln 2}$$

since $\int e^{x\ln 2}dx = \frac{1}{\ln 2}\int e^u du$ where $u = x\ln 2$ (so $du = (\ln 2)dx$).
Therefore, answer is **B**.

2. The points of intersection are (1,1), (0,0), and (-1,-1). The area in question is

$$\int_{-1}^{0} |x^3 - x^{1/3}|dx + \int_0^1 |x^3 - x^{1/3}|dx = 2\int_0^1 x^{1/3} - x^3 dx = 2(3/4 - 1/4) = 1$$

so that the correct answer is **A**.

3. First observe that

$$\tan x \sin^2 x \cos^3 x = \frac{\sin x}{\cos x}\sin^2 x\cos^3 x = \sin^3 x \cos^2 x = (1-\cos^2 x)\cos^2 x \sin x$$

so that making the substitution $u = \cos x$, so that $du = -\sin x\, dx$, the integral in question becomes

$$-\int (1-u^2)u^2 du = -\int u^2 - u^4 du = -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C$$

as $u = \cos x$, the answer is

$$-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$$

so choice **D** is correct.

4. Note that $\int x^{-2}dx = -\frac{1}{x} + C$, and $\int e^{-2x}dx = -\frac{1}{2}e^{-2x} + C$, so that

$$A = \int_1^\infty x^{-2}dx = \left(\lim_{u\to\infty}\left(\frac{-1}{u}\right)\right) - \left(-\frac{1}{1}\right) = 1$$

$$B = \int_0^\infty e^{-2x}dx = \left(\lim_{u\to\infty} -\frac{1}{2}e^{-2u}\right) - \left(-\frac{1}{2}e^{-2\cdot 0}\right) = \frac{1}{2}$$

Hence, $A - B = 1/2$, and the correct answer is **B**.

5. Correct answer is **C**.

$$F'(x) = (x^3)^3\ln(x^3)(x^3)' - x^3\ln(x)(x') = x^9(3\ln x)(3x^2) - x^3\ln x = x^3(9x^8-1)\ln x$$

6. Make the substitution $u = 1 - e^x$, so $du = -e^x dx$. Then the integral becomes

$$-\int u^{1/2}du = -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}(1-e^x)^{3/2} + C$$

so that **B** is correct.

Integration, Calculus level

7. $\int_{-2}^{7} \frac{dx}{x}$ converges $\iff$ both $\int_{-2}^{0} \frac{dx}{x}$ and $\int_{0}^{7} \frac{dx}{x}$ converge, in which case the first integral is the sum of the latter two. But

$$\int_{-2}^{0} \frac{dx}{x} = \lim_{x \to 0^-} \int_{-2}^{x} \frac{dx}{x} = \lim_{x \to 0^-} (\ln|x| - \ln 2) \Rightarrow \infty$$

so that this integral, and hence the original one, diverges. Therefore, **D**.

8. $f(1-x) = f(x) + 1 - 3x \Rightarrow f(1/2) = f(1/2) - 1/2 \Rightarrow 0 = -1/2$
This absurdity implies no such f exists; therefore, answer is **A**.

9. We apply integration by parts to see that

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{x^2+1} dx$$

In the final integral, make the substitution $u = x^2 + 1$; so $du = 2x\, dx$, so that

$$\int \frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln(x^2+1) + C$$

where the absolute value signs have been omitted since $x^2 + 1 > 0$ $(\forall x \in \mathbb{R})$.
Therefore the integral in question is

$$x \tan^{-1} x - \frac{1}{2} \ln(x^2+1) + C$$

so that choice **C** is correct.

10. Use the "shell method." Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of our interval, and define for $1 \le k \le n$, $t_k = (x_k + x_{k-1})/2$. Then, with ΔV the portion of the solid between the (revolved) lines $x = x_{k-1}$ and $x = x_k$,

$$\Delta V_k \approx \pi f(t_k)[(x_k+1)^2 - (x_{k-1}+1)^2] = 2\pi(t_k+1) f(t_k) \Delta x_k$$

So the volume is

$$\sum_{k=1}^{n} \Delta V_k \approx \sum_{k=1}^{n} 2\pi(t_k+1) f(t_k) \Delta x_k$$

which is a Riemann sum for $2\pi(x+1)f(x)$. Therefore,

$$V = 2\pi \int_{2}^{3} (x+1) f(x) dx = 2\pi \int_{2}^{3} (x+1)((x-2)^2+1) dx$$

and the answer is **D**.

11. Observe that

$$\int_{0}^{2\pi} \sqrt{1 - \cos 2x} dx = \int_{0}^{2\pi} \sqrt{1 - (1 - 2\sin^2 x)} dx = \sqrt{2} \int_{0}^{2\pi} |\sin x| dx$$

Now

$$\int_{0}^{2\pi} |\sin x| dx = \int_{0}^{\pi} \sin x \, dx + \int_{\pi}^{2\pi} -\sin x \, dx$$

Integration, Calculus level

Therefore, since cos′ = sin, we have

$$\int_0^{2\pi} |\sin x| dx = -\cos\pi - (-\cos 0) + \cos 2\pi - \cos\pi = 1 - (-1) + 1 - (-1) = 4$$

So that the value of the original integral is $4\sqrt{2}$ – hence **E**.

12. Let $p(t), v(t), a(t)$ be the position, velocity, and acceleration of the ball at the time t seconds after it is thrown down. As $a(t) = -32$, $v(t) = -32t + C$ – as $v(0) = 0$, we see $C = -28$, so $v(t) = -32t - 28$, and hence $p(t) = -16t^2 - 28t + C$, and as $p(0) = 144$, $p(t) = -16t^2 - 28t + 144$. By the quadratic formula, the ball hits the ground ($p(t) = 0$) at $t = 9/4$ (the other root being < 0). Hence the mean height of the ball is

$$\frac{\int_0^{9/4} (-16t^2 - 28t + 144)dt}{9/4}$$

Evaluating, we see this equals

$$\frac{-(16/3)(9/4)^3 - 14(9/4)^2 + 144(9/4)}{9/4} = 85.5$$

Therefore, answer is **B**.

13. Consider the partition $P = \{x_0, x_1, \ldots, x_n\}$, where $x_k = k/n$, $0 \le k \le n$. The sum in question is the left Riemann sum for

$$\int_0^1 (2x+3)^3 dx$$

with the partition (of [0,1]) above. Hence, as $n \to \infty$, the sum goes to the value of the integral above, which is

$$\frac{1}{8}(2 \cdot 1 + 3)^4 - \frac{1}{8}(2 \cdot 0 + 3)^4 = 68$$

In fact,

$$\sum_{i=0}^{n-1} \frac{1}{n}\left(\frac{2i}{n} + 3\right)^3 = 68 - \frac{49}{n} + \frac{8}{n^2}$$

So answer is **A**.

14. The standard way to do this is to integrate by parts twice. We use the method of undetermined coefficients. Let

$$f(x) = Ae^{2x}\cos(3x) + Be^{2x}\sin(3x)$$

Differentiate to see

$$f'(x) = (2B - 3A)e^{2x}\sin(3x) + (2A + 3B)e^{2x}\cos(3x)$$

And solve $2B - 3A = 0, 2A + 3B = 1$. This gives $A = 2/13, B = 3/13$, so answer (up to the constant) is

$$f(x) = \frac{2}{13}e^{2x}\cos(3x) + \frac{3}{13}e^{2x}\sin(3x)$$

which is **D**.

Integration, Calculus level

15. In the general case, Simpson's rule says:

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$$

Here $n = 10$, $f(x) = \sin x^2$, $x_i = 1 + 9i/10$, $0 \le i \le 10$, and the approximation resulting is (to three decimal places) 1.162, choice **A**.

16. Solving, $y = (9 - x^{2/3})^{3/2}$. Thus

$$y' = \frac{3}{2}\left(9 - x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right)$$

which means

$$(y')^2 = 9x^{-2/3} - 1$$

and the arclength equals

$$\int_0^{27} \sqrt{1 + [(9x^{-2/3} - 1)\quad]}dx = 3\int_0^{27} x^{-1/3}dx = \frac{9}{2}(27)^{2/3} = \frac{81}{2}$$

(where the upper limit is determined by solving $9 - x^{2/3} = 0$). Therefore, answer is **C**.

17. By symmetry, this is the same area as the surface obtained by revolving the region in question about the x-axis. This is

$$S = \int_0^{27} 2\pi(9 - x^{2/3})^{3/2}\sqrt{1 + [9x^{-2/3} - 1]}dx$$

which equals

$$\int_0^{27} 2\pi(9 - x^{2/3})^{3/2}3x^{-1/3}dx = 6\pi\int_0^{27}(9 - x^{2/3})^{3/2}x^{-1/3}dx$$

Make the substitution $u = 9 - x^{2/3}$, so $du = (-2/3)x^{-1/3}dx$. Then the integral in question becomes

$$6\pi\int_9^0 u^{3/2}\left(\frac{-3}{2}\right)du = 9\pi\int_0^9 u^{3/2}du = \frac{18\pi}{5}9^{5/2} = \frac{4374\pi}{5}$$

So answer is **D**.

18. One "leaf" is generated as θ runs from $-\pi/4$ to $\pi/4$, since the solutions of $\cos 2\theta = 0$ are $k\pi/4$, where k runs over the odd integers. The area enclosed by one leaf is thus

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2(2\theta)d\theta$$

Now

$$\int \cos^2(2\theta)d\theta = \int \frac{1+\cos(4\theta)}{2}d\theta = \frac{\theta}{2} + \frac{\sin 4\theta}{8} + C$$

so that the area is

$$\frac{1}{2}\left[\frac{\pi}{8} + \frac{\sin\pi}{8} - \left(-\frac{\pi}{8} + \frac{\sin(-\pi)}{8}\right)\right] = \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{8}$$

and the answer is **B**.

19. Apply Newton's law of cooling. We first solve the separable differential equation

$$\frac{dy}{dt} - ky = ky_0$$

(here $y(t)$ is temperature at time t, y_0 is the temperature of the surrounding medium) General solution is

$$y = y_0 + Ce^{kt} = 5 + Ce^{kt}$$

and, letting, $t = 0$, we see $C = 95$. At $t = 10$, we have

$$50 = 5 + 95e^{10k}$$

so that $k = \ln(45/95)/10$. Now solve

$$20 = 5 + 95e^{\ln(45/95)t/10}$$

One finds

$$t = \frac{\ln\left(\frac{15}{95}\right)(10)}{\ln\left(\frac{45}{95}\right)} \approx 24.7$$

So (to the nearest minute) 15 additional minutes are required, so answer is **C**.

20. Make the substitution $u = -x$, so $du = -dx$.

$$\int_{2}^{7} f(-x)dx = -\int_{-2}^{-7} f(t)dt$$

Now notice that

$$-\int_{-2}^{-7} f(t)dt = -\left[\int_{-2}^{-4} f(t)dt + \int_{-4}^{-8} f(t)dt + \int_{-8}^{-7} f(t)dt\right] = -(8 + 3 + -6) = -5$$

So that **B** is correct.

21. Clearly $\Theta = \ln 13$. Now

$$M - A = \int_{0}^{1} \frac{64 - 64x^2 - e^{2x}}{8\sqrt{1-x^2} + e^x} dx = \int_{0}^{1} 8\sqrt{1-x^2} - e^x dx$$

Also, since the area of the unit circle is π,.

$$\int_{0}^{1} 8\sqrt{1-x^2} = 8\frac{\pi}{4} = 2\pi$$

and

$$\int_{0}^{1} e^x dx = e - 1$$

so the final answer is $2\pi + 1 - e - \ln 13$, choice **D**.

Integration, Calculus level

22. Make the substitution $u = \tan x$, $du = \sec^2 x = u^2 + 1\, dx$. Consider

$$\int_0^1 \frac{1}{1+u}\frac{1}{1+u^2}du = \int_0^1 \frac{1/2}{1+u}du + \int_0^1 \frac{-u/2+1/2}{1+u^2}du$$

Now clearly

$$\int_0^1 \frac{1/2}{1+u}du = \frac{1}{2}\ln(1+1) - \frac{1}{2}\ln(1+0) = \frac{\ln 2}{2}$$

and

$$\int_0^1 \frac{-u/2+1/2}{1+u^2}du = \frac{-1}{2}\int_0^1 \frac{u}{1+u^2}du + \frac{1}{2}\int_0^1 \frac{du}{1+u^2} = \frac{-1}{4}\ln(1+1^2) + \frac{1}{2}\tan^{-1} 1 = \frac{-\ln 2}{4} + \frac{\pi}{8}$$

Adding, the answer is $\frac{\ln 2}{4} + \frac{\pi}{8} = \frac{2\ln 2+\pi}{8}$, choice **D**.

23. Use Pappus' Theorem. The center of mass is the average of the vertices, which works out to (6, 9/2). By "shifting" (3,3) – i.e., vertex A – to the origin, we see the area of the parallelogram is $\left|\det\begin{bmatrix} 1 & 6 \\ 2 & 3 \end{bmatrix}\right| = 9$. The distance from (6,9/2) to the line $y = -2x - 1$ is $\frac{2\cdot 6+9/2+1}{\sqrt{5}} = \frac{7\sqrt{5}}{2}$.
So by Pappus' Theorem, the volume is

$$2\pi\frac{7\sqrt{5}}{2}9 = 63\pi\sqrt{5}$$

So correct answer is **B**.

24. Correct answer is **C**. Certainly iii) is not sufficient, since the *Dirichlet function*

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ otherwise} \end{cases}$$

is not integrable on [0,1] (although it is clearly bounded). For every interval induced by a partition would contain both rational and irrational numbers, forcing the upper integral to be 1 and the lower integral to be 0. For proofs that i, ii, and iv are sufficient, see *Advanced Calculus*, by Paul C. DuChateau (HARPERCOLLINS, 1992).

25. Answer is **B**. Certainly f/g is not necessarily integrable (let f be identically 1, g identically 0) on [0,1]. Neither is $f \circ g$, for take

$$g(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \text{ is irrational} \\ 1/n \text{ if } x = m/n \text{ (lowest terms)} \in \mathbb{Q} \end{cases}$$

and

$$f(x) = \begin{cases} 0 \text{ if } x = 0 \\ 1 \text{ if } x \neq 0 \end{cases}$$

Then $f \circ g$ is the Dirichlet function mentioned in the solution to problem 22, which is NOT Riemann integrable on [0,1]. To see g is integrable on [0,1], notice the points of discontinuity are exactly the rational numbers in [0,1], so that f is integrable by

Theorem (Lebesgue). *Suppose $f(x)$ is defined and bounded on $I = [a,b]$. Then f is Riemann integrable on I if and only if the set of points on I where f is not continuous has measure zero.*

For proofs that fg and $f+g$ are necessarily integrable, see the aforementioned book by DuChateau.

Integration, Calculus level

26. Without loss of generality, let the board be a square with side length 1 centered at the origin, and let A be the top of the square. The condition that a point (x, y) be closer to A than to the center translates into $|y - 1/2| < \sqrt{x^2 + y^2}$. Squaring to remove the absolute value bars gives the equivalent condition

$$(y - 1/2)^2 < x^2 + y^2$$

which, upon rearranging, is equivalent to $y > 1/4 - x^2$. The probability a randomly chosen point in this square will satisfy this is the area of the region (since the area of the square is 1) inside the square and above the parabola $y = 1/4 - x^2$. This area is

$$\int_{-1/2}^{1/2} \frac{1}{2} - \left(\frac{1}{4} - x^2\right) dx = \int_{-1/2}^{1/2} \left(\frac{1}{4} + x^2\right) dx = \frac{1}{3}$$

So answer is **A**.

27. First notice that for fixed $k > 1$, $L_k = \lim_{n\to\infty} a_n$, where

$$0 \le a_n = \sum_{i=1}^{3n} \frac{1}{(2n+i)^k} \le \sum_{i=1}^{\infty} \frac{1}{(2n+i)^k} = \zeta(k) - \sum_{j=1}^{2n} \frac{1}{j^k}$$

Now as $n \to \infty$, clearly $\zeta(k) - \sum\limits_{i=1}^{2n} \frac{1}{i^k} \to 0$, so $L_k = 0$ for $k > 1$. Now

$$L_1 = \lim_{n\to\infty} \sum_{i=1}^{3n} \frac{1}{(2n+i)} = \lim_{n\to\infty} \sum_{i=1}^{3n} \left(\frac{1}{2 + i/n}\right) \frac{1}{n}$$

The latter sum is a Riemann sum for

$$\int_0^3 \frac{1}{2+x} dx = \ln \frac{5}{2} \approx 0.916$$

So answer is **B**.

Alternately, a solution can be derived from noting $\sum\limits_{k=1}^{n} \frac{1}{k} = \log n + \gamma + O\left(\frac{1}{n}\right)$, where γ is Euler's constant.

28. A regular hexagon can be thought of as six equilateral triangles stuck together. The hexagons here have side length $2y = \frac{2\sqrt{36-4x^2}}{3}$, so the cross-sectional area is

$$A = 6\frac{s^2\sqrt{3}}{4} = \frac{3s^2\sqrt{3}}{2} = 3\frac{4(36-4x^2)}{9}\frac{\sqrt{3}}{2} = \frac{2\sqrt{3}}{3}(36 - 4x^2)$$

Hence the volume in question is

$$\int_{-3}^{3} \frac{2\sqrt{3}}{3}(36 - 4x^2)\, dx = \frac{16\sqrt{3}}{3} \int_0^3 9 - x^2 dx = 96\sqrt{3}$$

and the answer is **A**.

Integration, Calculus level

29. Let (x, y, z) be the center of mass of the solid in problem 27. By symmetry, $x = y = 0$. To calculate the value of z, we note that this will be the mean value of the z-coordinate of the centers of mass of the hexagonal cross-sections. Now the height of the center of mass of the cross section is a, the length of the apothem. Using the relationship between the sides of a 30-60-90 right triangle, or by recalling from elementary geometry that (for our regular hexagon) we must have

$$\frac{3s^2\sqrt{3}}{2} = \frac{1}{2}a(6s)$$

we discover $a = \frac{s\sqrt{3}}{2}$. Now we calculate the mean value:

$$\frac{1}{3-(-3)}\int_{-3}^{3}\frac{2y\sqrt{3}}{2}dx = \frac{\sqrt{3}}{6}\int_{-3}^{3}\frac{\sqrt{36-4x^2}dx}{3} = \frac{2\sqrt{3}}{9}\int_{0}^{3}\sqrt{9-x^2}dx = \frac{\pi\sqrt{3}}{2}$$

Hence the center of mass is $(0, 0, \frac{\pi\sqrt{3}}{2})$, and the correct answer is **D**.

30. By the method of partial fractions, we know there exist real numbers $a_0, a_1, \ldots, a_m$ with

$$\frac{1}{x(x-1)\ldots(x-m)} = \frac{a_0}{x} + \frac{a_1}{x-1} + \cdots + \frac{a_m}{x-m}$$

The above is equivalent to (upon clearing denominators)

$$1 = \sum_{j=0}^{m} a_j \prod_{\substack{k=0 \\ k\neq j}}^{m} (x-k)$$

Letting $x = i$, where $0 \le i \le m$, we see

$$1 = a_i i(i-1)(i-2)\ldots(i-(i-1))(i-(i+1))\ldots(i-m)$$

So that

$$1 = a_i i!(-1)^{m-i}(m-i)(m-(i-1))\ldots 1 = a_i i!(-1)^{m-i}(m-i)!$$

which implies

$$a_i = \frac{1}{i!(m-i)!(-1)^{m-i}} = \frac{(-1)^{m-i}}{(m-i)!i!}\frac{m!}{m!} = \frac{(-1)^{m-i}\binom{m}{i}}{m!}$$

Therefore, the integral in question is

$$\sum_{i=0}^{m} \frac{(-1)^{m-i}}{m!}\binom{m}{i}\ln|x-i| + C$$

so that the correct answer is **D**.