2008 Mu Alpha Theta February Regional
Precalculus Team Round-SOLUTIONS

1.) 3
2.) 9
3.) C
4.) $2\pi - 2e$
5.) $HAHA$ or $HAHA_{25}$
6.) $\frac{\sqrt{3}\pi}{8}$
7.) 2
8.) $\frac{\pi}{3}$
9.) $1 + \frac{\pi}{2}$
10.) 465
11.) I,III,IV
12.) 2011
13.) $\frac{3\pi}{2}$
14.) 2
15.) $\arctan 10^k$

1.) At the beginning: 1,2,3,4. Step 1: 4,1,2,3. Step 2: 2,1,4,3. Step 3: 3,1,4,2. Step 4: 3,4,1,2. Step 5: 3,2,1,4; 2,3,1,4; 2,3,4,1. Step 6: 3,4,1,2. Step 7: 1,4,3,2. Step 8: 3,4,1,2. Step 9: 3,2,1,4. The first person had the number 2 after step 2, so the answer is 3.

2.) A is achieved when the three lines are non-parallel. In this case, there are 7 regions. B is achieved when all three lines are the same. In this case, there are 2 regions. The answer is $7 + 2 = 9$.

3.) All of the functions intersect at the point $(0, 1)$ except C, which crosses the y-axis at the origin.

4.) Since $\frac{5}{2} < e < 3$, $0.333 < \frac{1}{3} < \frac{1}{e} < \frac{2}{5}$. So, A is true. $(1 - \frac{1}{4})^2 = \frac{9}{16} > \frac{1}{2}$. So, B is false. $0 < e < \pi$ so $\sin(e) > 0$. So, C is false. $3 < \pi < 3.2$, and $2.7 < e < 2.8$, so $0.2 < \pi - e < 0.5$, so $2 < \frac{1}{\pi - e} < 5$. So, D is true. The answer is $2\pi - 2e$.

5.) The graphs intersect once between each $k\pi$ and $(k + 1)\pi$, $k \in \mathbb{Z}$, so they intersect 435 times. $626 \cdot 435 = (625 + 1) \cdot (HA_{25}) = HAHA_{25}$. So, the answer is $HAHA$ or $HAHA_{25}$.

6.) There are three possible configurations we must check. The spheres might be tangent along edges, face diagonals, or along the body the diagonal. In case 1, $2r = s$. In case 2, $2r = \sqrt{2}s$. In case 3, $4r = \sqrt{3}s$. We must choose the smallest r to insure that the spheres do not intersect at more points than the tangency points. So, we must use case 3. There is a full sphere and 8 one-eighth spheres inside the cube, so $\frac{V_s}{V_c} = \frac{2\frac{4}{3}\pi r^3}{s^3} = \frac{8\pi r^3}{3(\frac{4}{\sqrt{3}}r)^3} = \frac{8\sqrt{3}\pi}{64} = \frac{\sqrt{3}\pi}{8}$.

7.) A$=\frac{1}{1 - \frac{i}{2}} = \frac{2}{2 - i}$.

B$=\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^{2^{4^{251}}} = i^{4^{251}} = 1^{251} = 1$.

$x^4 - 1 = (x^2 - 1)(x^2 + 1) = 0$, so $x = -1, 1, -i, i$. C$=|1 - (-1)| \cdot i = 2i$.
So, the answer is $\frac{2}{2-i}(2 - i) = 2$.

8.) $V_0 = 2 \cdot \frac{\pi}{3} = \frac{2\pi}{3}$. $V_1 = 2 \cdot (2\frac{\pi}{3} - \frac{\pi}{12}) = \frac{7\pi}{6}$. $V_2 = 2\pi$. $A = \frac{V_1 - V_0}{1 - 0} = \frac{\pi}{2}$. $B = \frac{V_2 - V_1}{2 - 1} = \frac{5\pi}{6}$. $B - A = \frac{\pi}{3}$.

9.) The line $y = |x|$ intersects $y^2 + x^2 - 2x = 0$ at $x = \pm 1$ and $x = 0$ since $2x^2 - 2x = 0 \Rightarrow x = 0, 1$. We can split the area into two: a triangle with base length 2 and height 1 and a semicircle of radius 1. Thus, the total area equals $1 + \frac{\pi}{2}$.

10.) $A = \lim_{x\to 5} \dfrac{x^2+2x+3}{3x+4} = \dfrac{38}{19} = 2$. $B = \frac{5\cdot 4\cdot 3}{3!} = \frac{60}{6} = 10$. $C = \frac{1}{R_{pump}-R_{drain}} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$. $D = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = \frac{31}{16}$. $8\cdot A\cdot B\cdot C\cdot D = 8\cdot 2\cdot 10\cdot \frac{3}{2}\cdot\frac{31}{16} = 465$.

11.) Since all the pencils are assumed to be identical, ABC must be an equilateral triangle. Thus, I is true, II is false, and III is true. Since $\bar{DB} = \bar{EB}$ and $\bar{FB} = \bar{GB}$, the two triangles are SAS congruent. Thus, IV is true. Since $m\angle FBE$ is not necessarily $\frac{\pi}{2}$, V is false. The answer is I,III,IV.

12.) $2\sum_{i=1}^{n}\left(\dfrac{i+1}{n}\right) = 2\dfrac{\frac{n(n+1)}{2}+n}{n} = n+3$. For $n = 2008$, the answer is $2008+3 = 2011$.

13.) $\log|A| > 0 \Rightarrow |A| > 1$. Expanding by minors along the third column, $|A| = \sin x(\sin x - \cos x) - 1(-1) = \sin^2 x - \sin x\cos x + 1$. We need to find all $0 \le x < 2\pi$ such that $|A| = 1$, so $\sin^2 x - \sin x\cos x = \sin x(\sin x - \cos x) = 0$. This occurs at $x = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}$. By simple sign checking we see that $|A| > 1$ for $\frac{\pi}{4} < x < \pi$ and $\frac{5\pi}{4} < x < 2\pi$. The total length of the intervals is $\frac{3\pi}{2}$.

14.) $f(x) = \cos x$, $g(x) = \sin x$. $f(g(0)) + (f(x)+g(x))^2 + g(f(\frac{\pi}{2})) - g(2x) = f(0) + g(0) + f(x)^2 + g(x)^2 = 1+0+1 = 2$.

15.) $k = \log\sin(x) - \log\cos(x) = \log\tan x$. $\tan x = 10^k$. $x = \arctan 10^k$.