February Regional Calculus Team Round Solutions

1. **A, C, F** B is not necessarily true. D is not necessarily true because the derivative does not have to be differentiable everywhere. E is not always true because the second derivative does not have to have the same sign as the first derivative.

2. **1** **A**: L'Hopital's rule. $\lim_{x\to 0}\frac{\sin x}{1}=0$.

   B: Definition of e.

   C: Another definition of e.

   D: Synthetic division yields x^2+2. Plugging in 3 gives us 11.

   $\frac{C}{B}+\frac{A}{D}=\frac{e}{e}+\frac{0}{11}$ **= 1**

3. $5+e^2$ **A**: Multiply the numerator and the denominator by $1+\cos x$.

   $$\int_{\pi/6}^{\pi/3}\frac{1+\cos x}{\sin^2 x}dx=\int_{\pi/6}^{\pi/3}\csc^2 x+\csc x\cot x dx=-\cot x-\csc x\Big|_{\pi/6}^{\pi/3}=2$$

   B: $\int_0^{e^2}dx=e^2$

   C: A simple u-sub is required for this problem, with $u=\cos x, du=-\sin x dx$. Thus, our integral is $\int -\frac{du}{u}=-\ln u$. Substituting back in for u, we get $-\ln\cos x\Big|_0^{\pi/3}=\ln 2$.

   D: Long division gives us $x+4+\frac{5}{x-2}$. We integrate this to get $\frac{9}{2}-5\ln 2$.

   $$\frac{A}{4}+B+5C+D=\frac{1}{2}+e^2+5\ln 2+\frac{9}{2}-5\ln 2=5+e^2$$

4. **1** Close inspection of the point reveals that it is on the parabola's axis of symmetry, so the closest points are equidistant from the vertex. One way to find B is to draw a normal line to the parabola at point A. Logically, this normal line will pass through (-2, 3). In point-slope form, this line is $y-3=\left(-\frac{1}{2A+4}\right)(x+2)$. To find B, let x = A and see that $y=3+\left(-\frac{1}{2(A+2)}\right)(A+2)=\frac{5}{2}$.

   The symmetry of the graph allows B = D. To find A + C, let $\frac{5}{2}=x^2+4x+3$. The sum of the x-values that satisfy this equation is -4. Therefore, $-4+5=1$.

5. $-\frac{25}{6}$ This problem is most easily solved by Cramer's Rule. Since we are minimizing the sum, the denominator determinant is not necessary to compute since it is a constant. Applying Cramer's rule to the three variables, we get $N_x=\begin{vmatrix}p & -2 & 1\\ p^2 & -1 & -2\\ 2p & 2 & 2\end{vmatrix}=6p^2+12p$, $N_y=\begin{vmatrix}1 & p & 1\\ 2 & p^2 & -2\\ 1 & 2p & 2\end{vmatrix}=p^2+2p$, and

   $N_z=\begin{vmatrix}1 & -2 & p\\ 2 & -1 & p^2\\ 1 & 2 & 2p\end{vmatrix}=-4p^2+11p$. Adding these numerators together, the result is $3p^2+25p$,

   whose minimum value is at $p=-\frac{25}{6}$.

6. **63** Grouping like terms, we get $\frac{dy}{(y+1)\ln(y+1)} = \frac{dx}{x\ln x}$. Integrating each side, we get $\ln\left|\ln(y+1)\right| = \ln\left|\ln x\right| + C$. We can further simplify this to $\ln(y+1) = C\left|\ln x\right|$. Plugging in our initial conditions, we find that $C = 2$. Therefore, $\ln(y+1) = 2\ln 8 = \ln 64$ so $y+1 = 64,\ y = 63$.

7. **9 A**: $\frac{4\ln(i+n) - 4\ln n}{i+n} = \frac{4\ln(n(\frac{i}{n}+1)) - 4\ln n}{n(\frac{i}{n}+1)} = \frac{4\ln\left(\frac{i}{n}+1\right)}{\frac{i}{n}+1}\frac{1}{n} = 4\int_0^1 \frac{\ln(x+1)}{x+1}dx = 2\left(\ln 2\right)^2$

   B: $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^2}{n(i^2+n^2)} = \frac{i^2}{n^3\left(\frac{i^2}{n^2}+1\right)} = \frac{\frac{i^2}{n^2}}{\frac{i^2}{n^2}+1}\frac{1}{n} = \int_0^1 \frac{x^2}{x^2+1}dx$

   $$\int_0^1 \frac{x^2}{x^2+1}dx = \int_0^1 1 - \frac{1}{x^2+1}dx = x - \tan^{-1}x\Big|_0^1 = 1 - \frac{\pi}{4}$$

   C: $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{i^2+n^2} = \frac{i}{n^2\left(\frac{i^2}{n^2}+1\right)} = \frac{\frac{i}{n}}{\frac{i^2}{n^2}+1}\frac{1}{n} = \int_0^1 \frac{x}{x^2+1}dx = \frac{\ln 2}{2}$

   D: $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{i^2+n^2} = \frac{n}{n^2\left(\frac{i^2}{n^2}+1\right)} = \frac{1}{\frac{i^2}{n^2}+1}\frac{1}{n} = \int_0^1 \frac{dx}{x^2+1} = \tan^{-1}x\Big|_0^1 = \frac{\pi}{4}$

8. **509** **A**: $2(0+5+15+25+15) = 120$ **B**: $2(5+15+25+15+5) = 130$

   C: $\frac{1}{2}(2)(0+2(5)+2(15)+2(25)+2(15)+5) = 125$ **D:** $2(2+10+25+20+10) = 134$

   $A+B+C+D = 509$

9. **-1** Given the range of $\arctan x$, the numerator is $\arctan x - \frac{\pi}{2} = \frac{\pi}{2} - \frac{\pi}{2} = 0$ and the denominator also approaches zero when x approaches infinity. Therefore, L'Hopital's rule is required. After L'Hopital's, the limit is $\lim_{x\to\infty}\frac{\frac{1}{x^2+1}}{-\frac{1}{x^2}} = \lim_{x\to\infty} -\frac{x^2}{x^2+1} = -1$

10. **-32** **A**: x = distance Pratik is from pole. y = length of shadow. By similar triangles, $\frac{15}{6}=\frac{x+y}{y}$.

Therefore, $2x=3y$ and $2\frac{dx}{dt}=3\frac{dy}{dt}$. So $\frac{dy}{dt}=2(3)(\frac{1}{3})=2$.

B: Let y be the height of the ball above the ground and x be the distance between Kristy and the shadow of the ball. The height of the ball is described by $h(t)=16-16t^2$ and its velocity by $v(t)=-32t$. By similar triangles, $\frac{y}{16}=\frac{x}{x+4}$. Half a second into the descent, y = 12 and x = 12. Cross multiplying the ratios and taking the derivative, we get

$\frac{dx}{dt}=\frac{dy}{dt}\left(\frac{x+4}{16-y}\right)=-16\left(\frac{16}{4}\right)=-64$.

C: After 6 seconds, y = 400, x = 300, and d = 500. $x^2+y^2=d^2$, so $x\frac{dx}{dt}+y\frac{dy}{dt}=d\frac{dd}{dt}$.

Solve for $\frac{dd}{dt}$ to get 30.

A + B + C = **–32**

11. $\frac{\ln 2}{2}$ $\quad A+B-C=\int_{\pi}^{2\pi}\frac{\cos x\sin x+\cos^2 x-1}{\sin^2 x+\frac{1}{2}\sin 2x-x}dx=\frac{1}{2}\int_{\pi}^{2\pi}\frac{2\cos x\sin x+2\cos^2 x-1-1}{\sin^2 x+\frac{1}{2}\sin 2x-x}dx$.

Clearly, the top is half the derivative of the bottom. $\frac{1}{2}\ln\left|\sin^2 x+\frac{1}{2}\sin 2x-x\right|\Big|_{\pi}^{2\pi}=\frac{\ln 2}{2}$

12. **54** If x is the length of each of the squares, the volume of the box is $V=x(9-2x)(9-2x)$. We find the maximum by taking the derivative and setting it equal to zero. Solving for $12x^2-72x+81=0$, the two roots are 1.5 and 4.5. The side length can't be 4.5, so the volume of the box is $1.5(6^2)=54$.

13. **22** Let A=x and B=y on the coordinate plane. The "dartboard" in question is a square with vertices (0,0), (10,0), (10,10), and (0,10). Winning $20 entails the "dart" to be above the curve $xy=50$ and inside the square. The hyperbola lies inside the square between x =5 and x = 10. Therefore, the area corresponding to NOT winning is the area of the rectangle left of the hyperbola, 50, plus the area under the hyperbola, $\int_5^{10}\frac{50}{x}dx=50\ln 2$. Thus, the probability of NOT winning is $\frac{50+50\ln 2}{100}=\frac{1+\ln 2}{2}$ and thus the probability of winning is $1-\frac{1+\ln 2}{2}=\frac{1-\ln 2}{2}$. Thus, the expected value is $20\left(\frac{1-\ln 2}{2}\right)-6\left(\frac{1+\ln 2}{2}\right)=7-13\ln 2$.

14. **–4** Line N has slope of -1 and goes through the point (-1, -3). The equation for this line is $y = -x - 4$. Setting this equal to the parabola, we see that the line intersects the parabola at the points $R(-1,-3)$ and $S(-3,-1)$.

15. **3** The slope m is equal to $\tan\theta$, where θ is the angle L makes with the positive x axis. $y - 1 = (\tan\theta)(x-1)$ is L in point slope form. The x-intercept is $\dfrac{\tan\theta - 1}{\tan\theta}$ and the y-intercept is $-\tan\theta + 1$. The area is $A = \dfrac{1}{2}\left(\dfrac{\tan\theta - 1}{\tan\theta}\right)\left(1 - \tan\theta\right) = \dfrac{-\tan\theta}{2} + 1 - \dfrac{\cot\theta}{2}$. Taking the derivative, we get $\dfrac{dA}{dt} = \left(\dfrac{-\sec^2\theta}{2} + \dfrac{\csc^2\theta}{2}\right)\dfrac{d\theta}{dt}$. Since $\dfrac{d\theta}{dt} = -\dfrac{9}{4}$, $\dfrac{dA}{dt} = 3$