2005 Palm Harbor February Invitational
Geometry Answer Key

Individual

1. B
2. D
3. C
4. D
5. C
6. D
7. B
8. B
9. A
10. E
11. D
12. C
13. D
14. C
15. B
16. B
17. E
18. D
19. C
20. C
21. D
22. C
23. C
24. A
25. C
26. C
27. A
28. B
29. C
30. A

Team

1. $\frac{145}{5184}$

2. $\frac{3025\pi}{4}$

3. $4\sqrt{15}$

4. 14.0

5. $9\pi^7 + \frac{\pi^4}{2} + 8\pi + 6\sqrt{3} + 5\sqrt{2} - 5$

6. 12

7. 1.38

8. 60

9. 270

10. 15

11. 868

12. $147\sqrt{3} - 49\pi$

13. 5

14. 14

15. 0.08 centimeters (100 minutes)

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1. **B**

   Applying the Angle Bisector Theorem, we have $\frac{BD}{DC} = \frac{AB}{AC}$. Substituting the appropriate numbers, $\frac{8}{4} = \frac{10}{AC}$, and $AC = 5$.

2. **D**

   $\sqrt{289} = 17$, so by the Triangle Inequality Theorem, this could not be the length of the third side. All the other answers comply with the Triangle Inequality Theorem.

3. **C**

   Choice A would imply B, and vice versa, but neither can be directly concluded from the given information. Only choice C must be true.

4. **D**

   Using the Exterior Angle Theorem, we have $3x = (x+15)+(x+5) = 2x+20$, and $x = 20$. Since the sum of the measures of the angles of any triangle is $180°$, we have $(x+15)+(x+5)+(3x-15)+y = 180$. Substitute 20 for *x*, rearrange, and solve for *y*. $105 + y = 180$, and $y = 75$.

5. **C**

   Only the expressly stated conditions can be used, so since neither of the two given statements said anything about things that are not green, we cannot conclude anything about those things. For example, a red ladybug could live on Gazzle I; the given conditions do not rule that out. Therefore, only statement II is true, and the answer is C.

6. **D**

   $AO = AB = 7$, since both are radii of circle O. Applying the Pythagorean theorem. $AB = \sqrt{7^2 + 7^2} = 7\sqrt{2}$. So, $AB \times OA \times OB = 343\sqrt{2}$.

7. **B**

   The triangle formed by the altitude of the cone, the slant height, and the radius at the base is a 30-60-90 right triangle. So, if the height is 10, then the base is $\frac{10}{\sqrt{3}} = \frac{10\sqrt{3}}{3}$, and the volume is equal to $\frac{1}{3}(10)(\pi)\left(\frac{10}{\sqrt{3}}\right)^2 \approx 349.1$

8. **B**

   Since the radius is $\frac{3}{2}$ times as big, and the volume remained constant, the height must be $\frac{4}{9}$ what it was (this is because the volume is directly proportional to the square of the radius, so if the radius increases by that factor, the height must decrease by the square of that factor to maintain constant volume). $\left(\frac{4}{9}\right)(18) = 8$.

9. **A**
One-half times the product of the length of any side and the altitude to that side is always equal to the area of the triangle. Thus the ratio of the longest altitude to the shortest altitude is equal to the ratio of the longest side to the shortest side, or $\frac{45}{20} = \frac{9}{4}$.

10. **E**
The intersection of the angle bisectors of a triangle is known as the incenter, because that point is the center of the inscribed circle of that triangle.

11. **D**
Since the measure of an arc subtended by an inscribed angle in a circle is twice the measure of the inscribed angle, we have $2 \times 80 = w + 50$, and $w = 110$. Similarly, we have $2x = 360 - (2 \times 80)$ and $x = 100$. The sum of the measures of the angles in a quadrilateral equals 360 degrees, so $y = 360 - (80 + 75 + 100) = 105$. Thus, $x + y + w = 100 + 110 + 105 = 315$.

12. **C**
Applying the Pythagorean Theorem twice, we have
$AD = \sqrt{AB^2 - BD^2} = \sqrt{25^2 - 15^2} = \sqrt{400} = 20$, and
$DC = \sqrt{BC^2 - BD^2} = \sqrt{17^2 - 15^2} = \sqrt{64} = 8$. Using the same theorem again,
$AC = \sqrt{AD^2 + CD^2} = \sqrt{20^2 + 8^2} = \sqrt{464} = \sqrt{16 \times 29} = 4\sqrt{29}$.

13. **D**
Using the theorem about intersecting secant segments, we have $3(x + 3) = 4(8 + 4)$. Solving, we get $x = \frac{48}{3} - 3 = 13$.

14. **C**
The triangle formed by the ladder, the tree, and the ground is a right triangle with the ladder as the hypotenuse. Using the Pythagorean Theorem, the distance from the tree the ladder is placed is $\sqrt{15^2 - 12^2} = \sqrt{81} = 9$.

15. **B**
The measure of arc ANR is 360 degrees minus the measure of arc AR whose central angle is 100, which is equal to 260.

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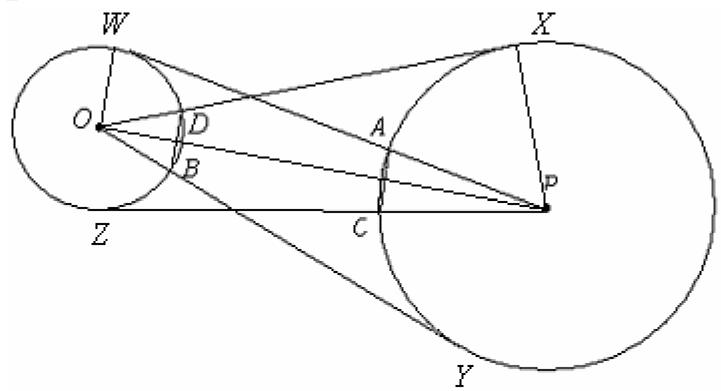
16. **B**



Since segments OX and PW are tangents to the circles, angles W and X are right angles. Let E be the midpoint of segment BD and let F be the midpoint of AC. Segment PO is the perpendicular bisector of both BD and AC. Then triangle EOD is similar to triangle XOP (share one angle, both have a right angle), and similarly (no pun intended) triangle FPA is similar to triangle WPO. Let $OW = OD = r$, and so $PX = PA = 2r$. We have $\frac{ED}{PX} = \frac{OD}{OP}$, and $\frac{ED}{2r} = \frac{r}{OP}$. Also, $\frac{AF}{OW} = \frac{AP}{OP}$, and $\frac{AF}{r} = \frac{2r}{OP}$. Thus, we have $AF = ED = (2r^2)(OP)$, and so we know $BD = AC$.

17. **E**

Choices A and B would result in a Side-Side-Angle relationship, which does NOT imply congruence. Choice C places the right angles and the corresponding hypotenuses and legs in the wrong relationships, and so does not imply congruence either. Thus, none of the given choices implies congruence.

18. **D**

The diagonal of the square is equal to the diameter of the circle, or twice the radius, $2r$. The radius of the circle is also $\frac{2}{3}$ the length of the altitude of the equilateral triangle; this is because the center of the circle is also the centroid of the equilateral triangle, and also the centroid of any triangle is $\frac{2}{3}$ of the way from the vertex to the midpoint of the opposite side. Thus, the altitude of the equilateral triangle is $\frac{3}{2}r$. The length of a side of the equilateral triangle is thus $\left(\frac{3}{2}r\right)\left(\frac{2}{\sqrt{3}}\right) = \sqrt{3}r$ (use the 30-60-90 right triangle formed by the altitude, a side, and half of the base). The formula for the area of an equilateral triangle is $\frac{s^2\sqrt{3}}{4}$, which in this case gives $\frac{(\sqrt{3}r)^2\sqrt{3}}{4} = \frac{3\sqrt{3}r^2}{4}$. The area of the square is $(2r)^2 = 4r^2$. Thus, the ratio of the area of the square to the area of the equilateral triangle is $\frac{4}{\frac{3\sqrt{3}}{4}} = \frac{16}{3\sqrt{3}} = \frac{16\sqrt{3}}{9}$.

19. **C**
From the given information, we know that D is the center of the circle, and so AR is a diameter. Therefore, ΔACR is inscribed in the semicircle, and $\angle ACR$ must be a right angle.

20. **C**
Euler's formula says that for any convex polyhedron, $V - E + F = 2$, where V is the number of vertices, E is the number of edges, and F is the number of faces. Using the given information, $E = 2V$ and $F = E - 4 = 2V - 4$. Thus, $V - 2V + 2V - 4 = 2$ and $V = 6$. Substituting this value for V yields $V + E + F = V + 2V + 2V - 4 = 6 + 2(6) + 2(6) - 4 = 26$.

21. **D**
The area swept out by the planet is equal to the area of a circle of radius $70{,}000{,}000 + 5{,}000$ miles minus the area of a circle with radius $70{,}000{,}000 - 5{,}000$ miles (the width of this annulus is equal to the diameter of the planet). Thus, this area equals $\pi(70{,}005{,}000^2 - 69{,}995{,}000^2) \approx 4.4 \times 10^{12}$.

22. **C**
Only choice C would produce a logical and useful result. The intersection of these two arcs is a point which is 7 units from one endpoint and 8 units from the other, thus forming the desired triangle.

23. **C**
Draw the two radii from the center of the circle to the endpoints of the side of the trapezoid with length 8. This is an isosceles triangle, and so the median from the center of the circle to the midpoint of the segment with length 8 is also an altitude. Thus, using the Pythagorean theorem, the height is equal to $\sqrt{6^2 - 4^2} = \sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$.

24. **A**
$BO = AO = 25$ since both are radii. The Pythagorean theorem gives us $OC = \sqrt{25^2 - 7^2} = \sqrt{576} = 24$. Thus, $BC = 25 - 24 = 1$, and $AB = \sqrt{7^2 + 1^2} = \sqrt{50} = 5\sqrt{2}$.

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25. **C**
Choices A, B, and D are either theorems or direct results of theorems. Choice C is false because lines A, B, and C could be noncoplanar.

26. **C**
Let *s* be the length of each side of the equilateral triangle, and let $x = BD = CF$. Then by the Pythagorean Theorem, $s^2 = 1^2 + x^2$ and $s = \sqrt{2}(1-x)$. Squaring this last equation, $s^2 = 2(1-x)^2 = 2x^2 - 4x + 2$. Thus, we have $1 + x^2 = 2x^2 - 4x + 2$, and rearranging, $x^2 - 4x + 1 = 0$. Using the Quadratic Formula, we get
$x = \frac{-(-4) \pm \sqrt{(-4)^2 - (4)(1)(1)}}{2(1)} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$. $2 + \sqrt{3} > 1$, so this is an extraneous solution. Substituting $x = 2 - \sqrt{3}$, we get
$s = \sqrt{2}(1 - (2 - \sqrt{3})) \approx 1.04$.

27. **A**
1) $2\pi r = 1$, and $r = \frac{1}{2\pi}$. The area is equal to $\pi r^2 = \pi\left(\frac{1}{2\pi}\right)^2 = \frac{1}{4\pi}$, which is irrational (and transcendental).
2) The area of any quadrilateral whose diagonals are perpendicular is equal to one-half the product of the lengths of the diagonals. The diagonals of a square are perpendicular and congruent, so the area of the square is equal to
$\left(\frac{1}{2}\right)(1)(1) = \frac{1}{2} = .5$, which is a terminating decimal.
3) The altitude to the hypotenuse of this isosceles right triangle is also the median to that side. Thus the length of the hypotenuse is $2(1) = 2$, and the area of the triangle is equal to $\left(\frac{1}{2}\right)(1)(2) = 1$, which is an integer.
4) Since the triangles are similar and the ratio of the lengths of the corresponding sides is $\frac{1}{3}$, the ratio of their areas is $\frac{1}{9}$. The area of the 3-4-5 right triangle is
$\left(\frac{1}{2}\right)(3)(4) = 6$, and so the area of the smaller triangle is $\left(\frac{1}{9}\right)(6) = \frac{2}{3} = .\overline{6}$, which is a repeating decimal.

28. **B**
Let *x* and z be the length and width of the surface of the pool, respectively. Then, the perimeter is $360 = 2x + 2z$ and $180 = x + z$. The volume of the pool can then be expressed as $(4)(x)(z) + \left(\frac{1}{2}\right)(8-4)(x)(z) = 6xz$. Substituting $z = 180 - x$, we have that the volume is equal to $(6)(x)(180 - x) = 1080x - 6x^2$. Graphing this function for the volume of the pool, we find that it opens downward, and so the

maximum value of the function is at the vertex. Putting this function into vertex form, we have $y - 48600 = -6(x^2 - 180x + 8100)$, and so the largest volume possible is 48600 cubic feet.

29. **C**
The sum of the area of the two isosceles triangles is equivalent to the area of a parallelogram with the same height and base. The base is the length of the side of the square, 6. Let *h* be the height. Then $6h + 6^2 = 60$, and $h = \frac{60-36}{6} = 4$. If we draw the altitude from the vertex to the side of the square of either triangle, we see that since the altitude is also the median, a 3-4-5 right triangle is formed. Therefore, the total fencing needed is equal to $5+5+6+6+5+5 = 32$.

30. **A**
The simplest way to do this problem is to find the distance between each person's house. Let *J* be Jenn's house, *T* be Taryn's house, *C* be Cody's house, and *H* be Channing's house. Then using the distance formula repeatedly, $JT = \sqrt{73}$, $JC = \sqrt{61}$, $JH = \sqrt{17}$, $TC = \sqrt{90}$, $TH = \sqrt{98}$, and $CH = \sqrt{20}$. Adding up the distances to each person's house, we find that the total distance everyone travels to Channing's house is smallest, roughly 18.5 units.