7. Hillsborough 3/14/92

Individual Calc. Test. Answers

#1. $\lim_{x\to -3} \frac{x^2+2x-8}{x^2+x-6} = \lim_{x\to -3} \frac{(x+4)(x-2)}{(x+3)(x-2)} = \lim_{x\to -3} \frac{x+4}{x+3}$ $\boxed{DNE}$

(D)

#2. Eq. for $\perp$ to grph. of $f(x)=x^3$ @ $(2,8)$.

$y-8 = -\frac{1}{y'}\Big|_{(2,8)}(x-2) \Rightarrow y-8 = -\frac{1}{3x^2}\Big|_{(2,8)}(x-2) \Rightarrow y-8 = -\frac{1}{12}(x-2)$

$\Rightarrow 12y-96 = -x+2 \Rightarrow \boxed{x+12y=98}$ (B)

#3. $\lim_{x\to -1} \frac{x^{1/2}+1}{x+1}$ DNE due to $-1 \notin \text{dom } x^{1/2}$. $\therefore$ NOTA is correct ans.

(E)

#4. $\lim_{x\to 0^+} \frac{3}{1+e^{-1/x}}$. as $x\to 0^+$, $\frac{1}{x}\to +\infty$, $e^{1/x}\to\infty$,

$\frac{1}{e^{1/x}} \to 0 \Rightarrow \lim_{x\to 0^+} \frac{3}{1+e^{-1/x}} = \boxed{3}$ (B)

#5. $f(t) = \text{Arcsin } t + \text{Arccos } t$, find $f'(\frac{\sqrt{2}}{2})$.

But $\text{Arcsin } t + \text{Arccos } t = \frac{\pi}{2}$ $\therefore f'(t)=0$ $\therefore \boxed{f'(\frac{\sqrt{2}}{2})=0}$ (A)

#6. $\lim_{x\to 0} (\cos x)^{1/x^2}$. $f(x) = (\cos x)^{1/x^2} \Rightarrow \ln f(x) = \frac{1}{x^2}\ln(\cos x)$

$\lim_{x\to 0}[\ln f(x)] = \lim_{x\to 0} \frac{\ln(\cos x)}{x^2} = \lim_{x\to 0} \frac{(1/\cos x)(-\sin x)}{2x} = \lim_{x\to 0}\left(-\frac{\tan x}{2x}\right)$ (C)

$= \lim_{x\to 0}\left(-\frac{\sec^2 x}{2}\right) = -\frac{1}{2}$ $\therefore \ln\left(\lim_{x\to 0} f(x)\right) = -\frac{1}{2}$ and $\boxed{\lim_{x\to 0} f(x) = e^{-1/2}}$

#7. $[f(x)=|x| \wedge x\neq 0] \Rightarrow \boxed{f'(x) = \frac{|x|}{x}}$ (B)

Pf. $f(x) = |x| = \sqrt{x^2} \Rightarrow f'(x) = \frac{1}{2}[x^2]^{-1/2}\cdot 2x = \frac{x}{\sqrt{x^2}} = \frac{x}{\sqrt{x^2}}\cdot\frac{\sqrt{x^2}}{\sqrt{x^2}}$

$= \frac{x\sqrt{x^2}}{x^2} = \frac{\sqrt{x^2}}{x} = \frac{|x|}{x}$

#8. $\int_1^4 \frac{(x+1)^2}{\sqrt{x}}\,dx = \int_1^4 (x^2+2x+1)\,x^{-1/2}\,dx = \int_1^4 (x^{3/2}+2x^{1/2}+x^{-1/2})\,dx$

$= \left[\frac{2x^{5/2}}{5} + \frac{4x^{3/2}}{3} + 2x^{1/2}\right|_1^4 = \left(\frac{2}{5}\cdot 32 + \frac{4}{3}8 + 4\right) - \left(\frac{2}{5}+\frac{4}{3}+2\right)$

$= \frac{2}{5}\cdot 31 + \frac{4}{3}\cdot 7 + 2 = \frac{3\cdot2\cdot31 + 5\cdot4\cdot7 + 2\cdot15}{15} = \frac{186+140+30}{15}$

$= \boxed{\frac{356}{15}}$ (C)

#9. $\int_{\frac{1}{e}}^{e} \frac{dx}{x(\ln x)^2}$. $\frac{1}{e} < 1 < e$ and $\ln 1 = 0 \Rightarrow$ this is an improper integral.

Consider $\int_M^e \frac{dx}{x(\ln x)^2}$ where $1 < M \le e$

Let $u = \ln x \Rightarrow du = \frac{dx}{x}$, $x = e \Rightarrow u = 1$
$x = M \Rightarrow u = \ln M = \bar{M}$

and $\int_M^e \frac{dx}{x(\ln x)^2} = \int_{\bar{M}}^1 \frac{du}{u^2} = -u^{-1}\Big|_{\bar{M}}^1 = -(1^{-1} - \bar{M}^{-1})$

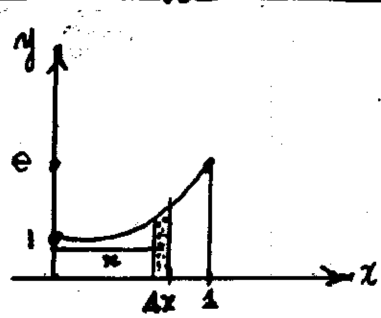
$= \bar{M}^{-1} - 1 = \frac{1}{\bar{M}} - 1 = \frac{1}{\ln M} - 1.$

And $\lim_{M\to1^+} \int_M^e \frac{dx}{x(\ln x)^2} = \lim_{M\to1^+}\left(\frac{1}{\ln M} - 1\right) = +\infty$ $\therefore$ $\boxed{\text{integral diverges}}$

(E) NOTA

#10. $\int_0^{\sqrt{3}} \frac{dx}{1+x^2} = \text{Arctan}\, x \Big|_0^{\sqrt{3}} = \boxed{\frac{\pi}{3}}$ (C)

#11.



$V = C\cdot A$; $C = 2\pi r = 2\pi x$; $A = f(x)\Delta x$
$= e^x \Delta x$

$\therefore V = 2\pi \int_0^1 x e^x\,dx$

Now $\int x e^x\,dx = \int u\,dv$ $u = x$, $dv = e^x dx$
$\Rightarrow du = dx$, $v = e^x$

$= xe^x - \int e^x dx = xe^x - e^x + C$

$\therefore V = 2\pi \left[xe^x - e^x\right|_0^1$

$= 2\pi\{(e - e) - (0 - 1)\} = \boxed{2\pi}$ (C)

(#12) $\displaystyle\int \frac{dx}{x^2-4}$

1st SOLN: $\displaystyle\frac{A}{x+2}+\frac{B}{x-2}=\frac{Ax-2A+Bx+2B}{x^2-4}=\frac{1}{x^2-4}$

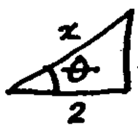
$$\Rightarrow \left.\begin{array}{l} A+B=0 \\ -2A+2B=1 \end{array}\right\} \Rightarrow \begin{array}{r} 2A+2B=0 \\ \underline{-2A+2B=1} \\ 4B=1 \end{array} \Rightarrow B=1/4 \Rightarrow A=-\tfrac{1}{4}$$

$$\Rightarrow \frac{1}{x^2-4}=\frac{1/4}{x-2}+\frac{-1/4}{x+2}$$

$$\Rightarrow \int\frac{dx}{x^2-4}=\frac{1}{4}\int\frac{dx}{x-2}-\frac{1}{4}\int\frac{dx}{x+2}=\frac{1}{4}\ln|x-2|-\frac{1}{4}\ln|x+2|+C$$

$$=\boxed{\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+C} \quad \text{C}$$

2nd SOLN:



$z=\sqrt{x^2-4}$. $\quad \frac{z}{2}=\tan\theta \Rightarrow z=2\tan\theta \Rightarrow z^2=4\tan^2\theta$

& $\frac{2}{x}=\cos\theta \Rightarrow 2\sec\theta=x \Rightarrow dx=2\sec\theta\tan\theta\, d\theta$

$$\therefore \int\frac{dx}{x^2-4}=\int\frac{2\sec\theta\tan\theta\, d\theta}{4\tan^2\theta}=\frac{1}{2}\int\frac{\sec\theta}{\tan\theta}\,d\theta=\frac{1}{2}\int\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}\,d\theta$$

$$=\frac{1}{2}\int\csc\theta\, d\theta=\frac{1}{2}\ln|\csc\theta-\cot\theta|+C\Rightarrow\frac{1}{2}\ln\left|\frac{x}{\sqrt{x^2-4}}-\frac{2}{\sqrt{x^2-4}}\right|+C=\frac{1}{2}\ln\left|\frac{x-2}{\sqrt{x^2-4}}\right|+C$$

$$=\frac{1}{2}\ln\left|\frac{\sqrt{x-2}\sqrt{x-2}}{\sqrt{x-2}\sqrt{x+2}}\right|+C=\frac{1}{2}\ln\left|\frac{\sqrt{x-2}}{\sqrt{x+2}}\right|+C=\frac{1}{2}\ln\left|\frac{x-2}{x+2}\right|^{1/2}+C=\boxed{\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+C}$$

(#13) $\displaystyle\int_0^{3/2}\frac{x\,dx}{\sqrt{1+2x}}$. $\quad u=1+2x\Rightarrow du=2dx$ and $u=1+2x\Rightarrow 2x=u-1 \Rightarrow x=\frac{u-1}{2}$

$x=0\Rightarrow u=1$; $x=\frac{3}{2}\Rightarrow u=4$

$$\int_0^{3/2}\frac{x\,dx}{\sqrt{1+2x}}=\int_1^4\frac{u-1}{2}\cdot\frac{1}{\sqrt{u}}\cdot\frac{du}{2}=\frac{1}{4}\int_1^4(u-1)u^{-1/2}\,du=\frac{1}{4}\int_1^4\left(u^{1/2}-u^{-1/2}\right)du$$

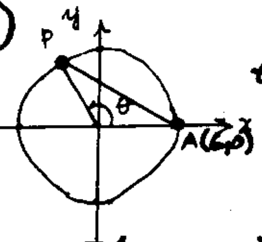
$$=\frac{1}{4}\left[\frac{2u^{3/2}}{3}-2u^{1/2}\right]_1^4=\frac{1}{4}\left\{\left(\frac{16}{3}-4\right)-\left(\frac{2}{3}-2\right)\right\}=\frac{1}{4}\left(\frac{4}{3}-\frac{-4}{3}\right)=\frac{1}{4}\left(\frac{8}{3}\right)=\boxed{\frac{2}{3}} \quad \text{A}$$

#14 $\lim_{n\to\infty}\left\{\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\dots+\sin\frac{n\pi}{n}\right)\frac{\pi}{n}\right\} = \lim_{n\to\infty}\sum_{i=1}^{n}\sin\left(0+\frac{i\pi}{n}\right)\frac{\pi}{n}$

$=\lim_{n\to\infty}\sum_{i=1}^{n} f(z_i)\Delta x_i$ [where $f(x)=\sin x$, $z_i=a+i\left(\frac{b-a}{n}\right)$, $\Delta x_i=\frac{b-a}{n}$, $a=0$, $b=\pi$]

$=\int_0^{\pi}\sin x\,dx = -\cos x\Big|_0^{\pi} = -(\cos\pi-\cos 0) = -(-1-1) = \boxed{2}$ — C

#15



$\theta = 2\pi\frac{t}{12} = \frac{\pi t}{6}$; $d^2 = \left[\left(6\cos\frac{\pi t}{6}-6\right)^2+\left(6\sin\frac{\pi t}{6}\right)^2\right]$

$= 36\left[\left(\cos\frac{\pi t}{6}-1\right)^2+\sin^2\frac{\pi t}{6}\right]$. Now @ $t=4$,

$d^2 = 36\left[\left(\cos\frac{2\pi}{3}-1\right)^2+\sin^2\frac{2\pi}{3}\right] = 36\left[\left(-\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right] = 36\left[\frac{12}{4}\right] = 36\cdot 3 \Rightarrow \underline{d=6\sqrt{3}}$

Next: $2dd' = 36\left\{2\left(\cos\frac{\pi t}{6}-1\right)\cdot\left(-\sin\frac{\pi t}{6}\right)\cdot\frac{\pi}{6} + 2\sin\frac{\pi t}{6}\cdot\cos\frac{\pi t}{6}\cdot\frac{\pi}{6}\right\}$

$= 36\cdot 2\cdot\frac{\pi}{6}\left\{\sin\frac{\pi t}{6}\right\} = 12\pi\sin\frac{\pi t}{6}$

$\Rightarrow d' = \frac{6\pi}{d}\sin\frac{\pi t}{6} \rightarrow d'\Big|_{t=4} = \frac{6\pi}{6\sqrt{3}}\cdot\sin\frac{2\pi}{3} = \frac{\pi}{\sqrt{3}}\cdot\frac{\sqrt{3}}{2} = \boxed{\frac{\pi}{2}}$ — D

#16 $f(x) = \frac{2}{3}x^{3/2} \Rightarrow f'(x) = x^{1/2} \Rightarrow [f'(x)]^2 = x$.

$L_{[1,4]}f = \int_{x=1}^{x=4}\sqrt{1+[f'(x)]^2}\,dx = \int_{x=1}^{x=4}(1+x)^{1/2}dx = \frac{2}{3}(1+x)^{3/2}\Big|_{x=1}^{x=4}$

$= \boxed{\frac{2}{3}\left[5^{3/2}-2^{3/2}\right]}$ — D

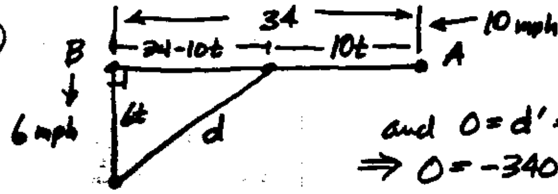
#17 D. $\boxed{\{\mathrm{I}, \mathrm{IV}, \mathrm{V}\}}$

#18 $f(x) = x$; $g(x) = \cos x + x$; $0 \le x \le 2\pi$. Find A. On $[0, \frac{\pi}{2}]$ $g(x) \ge f(x)$

$A_1 = \int_0^{\pi/2}[(\cos x + x) - x]\,dx = \int_0^{\pi/2}\cos x\,dx = \sin x\Big|_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 \Rightarrow A = 4A_1 = \boxed{4}$ D

#19 $f(x) = x^4 - 4x + 1 \Rightarrow f'(x) = 4x^3 - 4 \Rightarrow f''(x) = 12x^2$. $0 = f'(x) = 4(x^3-1) \Rightarrow x = 1$
$\wedge$ $f''(1) > 0 \Rightarrow f$ has a min. @ $x = 1$. Now for $\pm\varepsilon$ (small) $f''(\pm\varepsilon) > 0$, so f does not change concavity @ $x=0$, which is the soln. to $0 = f''(x) = 12x^2$.
$\therefore$ f has no p.o.i. ANS D

#20

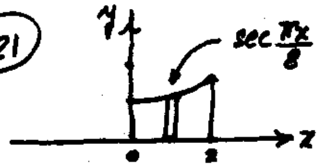


$d^2 = (34-10t)^2 + (6t)^2 \Rightarrow 2dd' = 2(34-10t)(-10) + 2(6t)(6)$

and $0 = d' \Rightarrow 0 = \cancel{2}(34-10t)(-10) + \cancel{2}(6t)(6)$

$\Rightarrow 0 = -340 + 100t + 36t \rightarrow t = \frac{340}{136} = \frac{170}{68}$ $\quad 68\overline{)170}$ = $\boxed{2.5}$ — A

#21

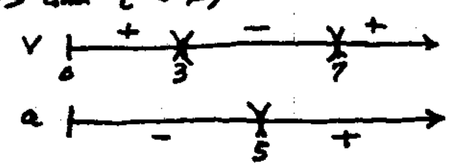


$A = \int_{x=0}^{x=2} \sec \frac{\pi x}{8}\, dx = \frac{8}{\pi} \int_{x=0}^{x=2} \sec \frac{\pi x}{8}\, d\left(\frac{\pi x}{8}\right) = \frac{8}{\pi} \ln\left|\sec \frac{\pi x}{8} + \tan \frac{\pi x}{8}\right|_0^2$

$= \frac{8}{\pi}\left\{ \ln\left|\sec \frac{\pi}{4} + \tan \frac{\pi}{4}\right| - \ln\left|\sec 0 + \tan 0\right| \right\} = \boxed{\frac{8}{\pi} \ln\left|\sqrt{2}+1\right|}$ — A

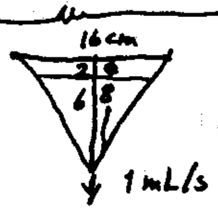
#22 Speed is decreasing when v and a have opposite signs.

$s = t^3 - 15t^2 + 63t \Rightarrow v = 3t^2 - 30t + 63 \Rightarrow a = 6t - 30$. $a > 0 \Rightarrow 6t > 30$
$\Rightarrow t > 5$. $v < 0 \Rightarrow t^2 - 10t + 21 < 0 \Rightarrow (t-3)(t-7) < 0 \Rightarrow (t < 3$ and $t > 7)$ or
$(t > 3$ and $t < 7)$

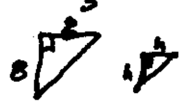


speed is decreasing on $\boxed{(0,3) \cup (5,7)}$ — C

#23



$V = \frac{1}{3}\pi r^2 h \Rightarrow V = \frac{1}{3}\pi h^3 \Rightarrow \frac{dV}{dt} = \pi h^2 \frac{dh}{dt}$



$= -1 = \pi(6)^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{-1}{36\pi}$

$\therefore$ rate that water level is <u>dropping</u> is $\boxed{\frac{1}{36\pi} \text{ cm/s}}$ — A

#24 $f'(x) = 3x^{1/2} + 7 \Rightarrow f(x) = \frac{2 \cdot 3}{3} x^{3/2} + 7x + C = 2x^{3/2} + 7x + C$ and
$48 = f(4) = 2(4)^{3/2} + 28 + C = 16 + 28 + C = 44 + C \Rightarrow C = 4$
$\Rightarrow \boxed{f(x) = 2x^{3/2} + 7x + 4}$ — B

#25 $V = s^3 \Rightarrow dV = 3s^2 ds \wedge ds = 0.004s \Rightarrow dV = 3s^2(0.004s) = \boxed{0.012 s^3}$ — A

#26 $F(x) = \int_3^{x^2} t(t^2+2)^{1/2} dt$, $D_x F(x) = D_x\left(\int_3^{g(x)} f(t)\, dt\right) = f(g(x))\, g'(x)$

$= x^2(x^4+2)^{1/2} \cdot 2x = \boxed{2x^3(x^4+2)^{1/2}}$ — C

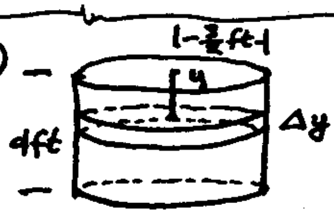
#27 Av. val of $f(x)=x^2$ over $[1,4]$

$$\text{Av.} = \frac{1}{4-1}\int_1^4 x^2\,dx = \frac{1}{3}\left[\frac{x^3}{3}\right|_1^4 = \frac{1}{9}\left(x^3\right|_1^4 = \frac{1}{9}(64-1) = \boxed{7} \text{ — B}$$

#28 $f(x)=\ln x \wedge g(x)=x^2+2x \Rightarrow f\circ g(x) = f(g(x)) = \ln(x^2+2x)$

$$= \ln[x(x+2)] = \boxed{\ln x + \ln(x+2)} \text{ — A}$$

#29



$\Delta Wt = 62\cdot\pi\left(\frac{3}{2}\right)^2\Delta y$

$\Delta W = 62\pi\left(\frac{3}{2}\right)^2 y\,\Delta y$

$$\begin{array}{r} 124 \\ \times 9 \\ \hline 1116 \end{array}$$

C

$$W = 62\pi\left(\frac{3}{2}\right)^2\int_0^4 y\,dy = 62\pi\,\frac{9}{4}\cdot\frac{1}{2}\left[y^2\right|_0^4 = \frac{62\pi 9}{\cancel{4}_1\cdot\cancel{2}_1}\,\cancel{16}^{\,2} = \boxed{1116\pi \text{ ft-lbs}}$$

#30 $y = x^x \Rightarrow \ln y = x\ln x \Rightarrow D_x(\ln y) = D_x(x\ln x)$

$$\Rightarrow \frac{1}{y}\cdot y' = x\cdot\frac{1}{x} + \ln x = 1+\ln x \Rightarrow y' = y(1+\ln x) = \boxed{x^x(1+\ln x)} \text{ — D}$$