Precalculus Team Questions

FAMAT Regional Competition, Jan 23, 1993 by Mike Galpin

1.) $\sin(A) = \frac{3}{5}$, $\cos(A) = \frac{-4}{5}$ (since $A \in Q\,II$)

$\cos(B) = \frac{7}{25}$, $\sin(B) = \frac{-24}{25}$ (since $B \in Q\,IV$)

$\cos(B-A) = \cos(B)\cos(A) + \sin(B)\sin(A) = \frac{-28-72}{5\cdot 25} = \frac{-100}{5\cdot 25} = -\frac{4}{5} = \frac{m}{n}$

$|m| + |n| = \boxed{9}$

2.) # ways of picking one man = 5

ways of picking one woman = 3

ways of picking 2 others = $6\cdot 5$

$5\cdot 3\cdot 6\cdot 5 = \boxed{450}$

3.) $420 = 2^2 3^1 5^1 7^1$

$A = \tau(420) = (2+1)(1+1)(1+1)(1+1) = 3\cdot 2^3 = 24$

(see #22) $B = \sigma(420) = \left(\frac{2^{2+1}-1}{2-1}\right)\left(\frac{3^{1+1}-1}{3-1}\right)\left(\frac{5^{1+1}-1}{5-1}\right)\left(\frac{7^{1+1}-1}{7-1}\right) = (7)(4)(6)(8) = 1344$

$C = \varphi(420) = 420\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right) = 420\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right) = 96$

$\frac{B}{AC} = \frac{7\cdot 2^6\cdot 3}{3\cdot 2^3\cdot 2^5\cdot 3} = \frac{7}{12} = \frac{m}{n} \Rightarrow m+n = \boxed{19}$

4.) $\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)} \Rightarrow \tan(y) = 0$

$y = 0, \pi \Rightarrow \boxed{\pi}$

5.) $Am(8,2) = \frac{8+2}{2} = 5$

$Gm(8,2) = \sqrt{8\cdot 2} = 4$

$Hm(8,2) = \frac{1}{2}\left(\frac{1}{8}+\frac{1}{2}\right)^{-1} = \frac{16}{5} \Rightarrow Am+Gm+Hm = 5+4+\frac{16}{5} = \frac{61}{5} = \frac{m}{n} \Rightarrow m+n = 66$

6.) To determine if an integer is divisible by 7, take the integer's last digit, remove it, multiply it by 2, and subtract from the rest.

Algorithmically,

$$r_n = q_n - 10\left\lfloor \frac{q_n}{10} \right\rfloor$$
$$q_n = \frac{q_{n-1} - r_{n-1}}{10} - 2r_{n-1}$$

Let q_0 be the number whose divisibility is in question, then if $7|q_0$ then $7|q_i$ for all i, So $q_0 = 1A323492110877$ can be reduced to the congruence:

$$101 + 10A + -14 \equiv 0 \pmod 7$$
$$3 + 3A \equiv 0 \pmod 7$$
$$A \equiv 6 \pmod 7$$

So since $1 \leq A \leq 9$, $\boxed{A = 6}$

7) A: $x = 0.1212\ldots$
$100x = 12.12\ldots$
$99x = 12$
$x = \frac{12}{99}$ which is $\left(\frac{8}{35}\right)_{10}$, so is unreducable
$\Rightarrow 12 + 55 = 67$

B: $\frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1 + r_2}{r_1 r_2} = \frac{\frac{-3}{5}}{\frac{1}{5}} = -3$

C: Length of median to hypot. $= \frac{1}{2}(\text{hypot.}) = 13$

D: $0+0+1+1+2+2+3+3+4+4+5 = 25$

$\log_2(A+B) + 2C - D = \log_2(67-3) + 26 - 25 = \log_2(64) + 1 = 6 + 1 = \boxed{7}$

8) A = The number of primes $\leq 93 = 24$ (Just count them)
$B = f(3-1) = 3^3 + 3^2 + 3 + 1 = 40$
C: $x^4 = 4 \Rightarrow x^4 - 4 = 0 \Rightarrow$ Product of roots $= -4$
D: $\frac{2\pi}{2\pi/5} = 5 \Rightarrow AD + BC = 120 - 160 = \boxed{-40}$

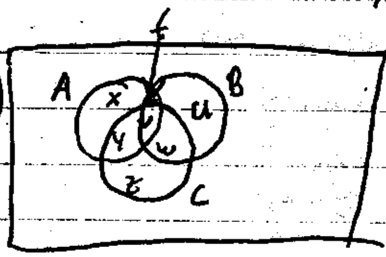
9) Supposed to get: $100x+y$ (in cents)
Got: $100y+x$

$\Rightarrow 100y+x-68 = 2(100x+y) \Rightarrow 98y-199x = 68$

$\Rightarrow 98y = 68+199x \Rightarrow y = 2x + \frac{68+3x}{98} \Rightarrow x=10$ is a solution

So $(10, 21)$ is a solution $\Rightarrow (10+199K, 21+98K)$ is a solution, but $y < 100 \Rightarrow (10, 21)$ is only solution
$\Rightarrow x+y = 10+21 = \boxed{31}$

10.)



(1) $x+y+v+t = 11$ $(|A|)$
(2) $y+v+w+z = 9$ $(|C|)$
(3) $y+v = 5$ $(|A\cap C|)$
(4) $t+v = 4$ $(|A\cap B|)$
(5) $v+w = 5$ $(|B\cap C|)$
(6) $v = 3$ $(|A\cap B\cap C|)$
(7) $x+y+v+t+w+z+u = 19$ $(|A\cup B\cup C|)$

$|B| = t+v+w+u = ?$

From (3), (4), (5) and (6) $\Rightarrow y=2, t=1, w=2$
From (1) $x=5$, from (2) $z=2$, and from (7) $u=4$
$|B| = t+v+w+u = 1+3+2+4 = \boxed{10}$

11.) Denote the Euler-Phi-Function of an integer n as $\varphi(n) =$ the number of integers less than and relatively prime to n, or in symbols

$$\Phi = \{m \mid (m,n)=1\} \quad (m \in \mathbb{Z}^+), (m < n)$$

$$\varphi(n) = |\Phi|$$

An easily provable theorem is, if $(a,n)=1$, then $a^{\varphi(n)} \equiv 1 \pmod{n}$
(For a proof see any Number Theory or Abstract Algebra Text)
So if $(n,10)=1$, then $n^{\varphi(10)} \equiv 1 \pmod{10} \Rightarrow n^4 \equiv 1 \pmod{10}$ $(\varphi(10)=4)$
$\Rightarrow n^{4\cdot 498+1} \equiv n \pmod{10}$. Also $2^n \equiv 2^{n+4} \pmod{10} \Rightarrow 2^{1993} \equiv 2 \pmod{10}$ and