FAMAT Regional Mathematics Competition
Calculus Team Answers
February 8, 1997

1. 343/6

2. Two

3. $\frac{2}{9}(1+x^3)^{1.5}+C$

4. $f(y) = \dfrac{y^2}{1-\ln(y)}$

5. $54\pi^2$

6. $\dfrac{\pi}{12}+\dfrac{\sqrt{3}}{2}-1$

7. 27/8

8. 14

9. $\sqrt{3}$

10. $54\sqrt{3}$

11. 120

12. $\dfrac{\pi}{8}-\dfrac{\sqrt{3}}{6}$

13. 3600

14. $2/3 < x < 10/3$

15. $\dfrac{1+y+2xy}{12x^2y-3x^3-3x^4}$

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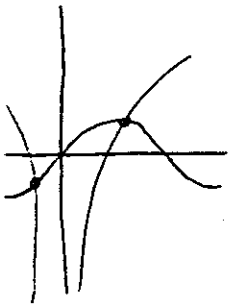
$\int_2^5 (10+3x-x^2)\,dx = \frac{343}{6}$



Two intersections

$x^3\sqrt{x^{-2}+x}\,dx = \int x^2\sqrt{1+x^3}\,dx$

$= \frac{2}{9}(1+x^3)^{3/2}+C$

$y = x^y \Rightarrow \ln y = y\ln x \Rightarrow \ln x = \frac{\ln y}{y}$

$\frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}\ln x + \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{y}{\frac{x}{y} - x\ln x}$

$\frac{y}{(\frac{1}{y})(1-\ln y)} \Rightarrow f(y) = \frac{y^2}{1-\ln y}$

Volume by Theorem of Pappus

$= 2\pi \cdot 3 \cdot 9\pi = 54\pi^2$

$u = \arcsin x \quad dv = 1 \quad du = \frac{1}{\sqrt{1-x^2}} \quad v = x$

$\int \sin x\,dx = x\arcsin x - \int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}}$

$x\arcsin x + \sqrt{1-x^2}\Big|_0^{1/2} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$

$x\frac{dy}{dx} + y = 2x + 2\frac{dy}{dx} \Rightarrow \frac{dx}{dy} = \frac{x-2}{2x-y}$

$\frac{d^2x}{dy^2} = \frac{(2x-y)\frac{dx}{dy} - (x-2)(2\frac{dx}{dy}-1)}{(2x-y)^2}$

at $(3,9)$, $dx/dy = -1/3$; $\frac{d^2x}{dy^2} = 8/27$

$\frac{d^2y}{d^2x} = \frac{1}{8/27} = 27/8$

$t = \int_0^{\pi} \sqrt{2}\sqrt{\frac{1}{2}(1+\cos x)}\,dx = \sqrt{2}\int_0^{\pi} \cos\frac{x}{2}\,dx = 2\sqrt{2}$

continued in next column.

B: $f'(x) = -\cos(\cos x^2)\cdot 2x\cdot\sin(x^2)$

$f''(x) = -2\sin(x^2)\cos(\cos x^2)$

$-2x\cdot\cos(x^2)\cdot 2x\cdot\cos(\cos x^2)$

$-2x\cdot\sin(x^2)\cdot(-\sin(\cos x^2))\cdot(-\sin x^2)(2x)$

$f''(\sqrt{\pi}/2) = -2$

$\int_{-2}^{2\sqrt{2}} (x^3+x)\,dx = 14$

9. $a^3+b^3+c^3 = ac^2+bc^2+c^3 \Rightarrow a^3+b^3 = ac^2+bc^2$

$\Rightarrow (a+b)(a^2-ab+b^2) = (a+b)c^2$

$a^2-ab+b^2 = c^2$

Law of cosines: $a^2 \quad 2ab\cos C + b^2 = c^2$

$\Rightarrow 2\cos C = +1 \Rightarrow C = \pi/3$. $\tan\pi/3 = \sqrt{3}$

10. This solid would be a sphere if the cross-sections were circles instead of hexagons.

$\Rightarrow \frac{A_{hexagon}}{A_{circle}} = \frac{V_{solid}}{V_{sphere}} \Rightarrow \frac{6\left(\frac{s^2\sqrt{3}}{4}\right)}{\pi s^2} = \frac{x}{\frac{4}{3}\pi\cdot 3^3}$

$\Rightarrow x = \frac{6\sqrt{3}}{4\pi}\cdot\frac{4}{3}\pi\cdot 3^3 = 54\sqrt{3}$

11. $\Gamma(n) = (n-1)! \Rightarrow \Gamma(6) = 5! = 120$

12. $f'(x) = \frac{(x+2)(2x\operatorname{arccsc} x + x^2\cdot\frac{-1}{x\sqrt{x^2-1}}) - x^2\operatorname{arccsc} x}{(x+2)^2}$

at $x=2$ $= \frac{\pi}{8} - \frac{\sqrt{3}}{6}$

13 A: 0 to 192 – all multiples of 4 $\Rightarrow 49$

B: ${}_8C_4 = 70$ C: $7!/2 = 2520$

$2520\cdot 70/49 = 3600$

14. $y' = 3x^2-4x-20 = (3x-10)(x+2)$

$\Rightarrow (-2, 10/3)$

$y'' = 6x-4 \Rightarrow x > 2/3$

$\Rightarrow (2/3, 10/3)$

15. $xy' + y + 2xy + x^2y' = 4y\cdot y' - 1$

$\Rightarrow \frac{dy}{dx} = \frac{+1+y+2xy}{-x \quad x^2+4y} \qquad \frac{dx^3}{dx} = 3x^2$

$\frac{d}{dx^3} = \frac{dy}{dx}\cdot\frac{dx}{dx^3} = \frac{1+y+2xy}{12x^2y-3x^4-3x^3}$