February 1999 FAMAT Regional Mathematics Competition
Calculus Individual Test Answer Key

1	D
2	D
3	B
4	C
5	E
6	E
7	D
8	C
9	B
10	D
11	B
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13	A
14	D
15	A
16	B
17	B
18	C
19	B
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21	B
22	C
23	D
24	B
25	C
26	A
27	D
28	D
29	B
30	A

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D 1. $3x^2+6x-9=0 \Rightarrow x=1,-3 \Rightarrow f(-4)=32, f(-3)=39, f(1)=7, f(2)=14$

D 2. $3x^2-12x-36>0 \Rightarrow (x-6)(x+2)>0 \Rightarrow (-\infty,-2)\cup(6,\infty)$
$6x-12>0 \Rightarrow x>2 \qquad \Rightarrow (6,\infty)$

B 3. $y'=\dfrac{-x^2-2x+1}{(x^2+1)^2} \Rightarrow x=-1\pm\sqrt{2} \Rightarrow y=\dfrac{1\mp\sqrt{2}}{2}$

C 4. $\dfrac{d}{dx}\left(x+2+\dfrac{1}{x}\right)=1-\dfrac{1}{x^2}=\dfrac{x^2-1}{x^2}$

E 5. Point of inflection is at $\dfrac{-b}{3a}=\dfrac{6}{-3}=-2$. $f'(-2)=-12+24-12=0$

E 6. $f'(x)=4(1)^3+12(1)^2-18(1)+2=0$. Equation is $y=-1$.

D 7. $A=\dfrac{1998!}{2}$, $B=1998!$, $C=1999\cdot1998\cdot1997$, $D=\dfrac{1999\cdot1998\cdot1997}{3\cdot2\cdot1}$. Answer $=2\cdot3\cdot2\cdot1=12$

C 8. $y=x^2$ is even, $x=y^3-y$ is not a function, and $x=y-3$ is neither odd nor even.
$y=x$, $y=\sin x$, $x=y^{1/3}$, and $y=\operatorname{Arcsin}x+x$ are odd.

B 9. $\dfrac{dV}{dt}=\dfrac{d}{dt}(s^3)=3s^2\dfrac{ds}{dt} \Rightarrow \dfrac{dS}{dt}=\dfrac{d}{dt}(6s^2)=12s\dfrac{ds}{dt}=\left(\dfrac{4}{s}\right)3s^2\dfrac{ds}{dt}=\left(\dfrac{4}{s}\right)\dfrac{dV}{dt}=2$

D 10. $y=x^{x^{x^{\cdot^{\cdot}}}}=x^y \Rightarrow \ln y=y\ln x \Rightarrow \left(\dfrac{dy}{dx}\right)\cdot\dfrac{1}{y}=\dfrac{y}{x}+\left(\dfrac{dy}{dx}\right)\cdot\ln x \Rightarrow \left(\dfrac{dy}{dx}\right)\cdot\dfrac{1}{2}=\dfrac{2}{\sqrt{2}}+\left(\dfrac{dy}{dx}\right)\cdot\ln\sqrt{2}$
$\Rightarrow \dfrac{dy}{dx}=\dfrac{2\sqrt{2}}{1-\ln 2}$.

B 11. $\dfrac{2x^3-a^2x^2-2a^2x+a^4}{ax^2-a^3}=\dfrac{(x-a)(2x^2+(2a-a^2)x-a^3)}{a(x-a)(x+a)}=\dfrac{(2x^2+(2a-a^2)x-a^3)}{a(x+a)}=\dfrac{4a^2-2a^3}{2a^2}=2-a$

E 12. $s=t^4-2t^3-3t^2+4t-12 \Rightarrow v=4t^3-6t^2-6t+4 \Rightarrow a=12t^2-12t-6 \Rightarrow \dfrac{da}{dt}=24t-12$.
Max acceleration at $t=\dfrac{1}{2}$. Change in direction $\Rightarrow$ velocity equals 0 $\Rightarrow t=-1,\dfrac{1}{2},2$. $\dfrac{1}{2}-\dfrac{1}{2}=0$

A 13. $-x^2+2x+1=x^2+2x-1 \Rightarrow x=\pm1 \Rightarrow \displaystyle\int_{-1}^{1}\left[(-x^2+2x+1)-(x^2+2x-1)\right]dx=\dfrac{8}{3}$

D 14. There are 13 intersections. 7 for $y>0$ and 6 for $y<0$.

A 15. $D=$ distance between them. $D^2=(10t+15t)^2+100^2 \Rightarrow \dfrac{dD}{dt}D=625t$
At $t=3 \Rightarrow D=125 \Rightarrow \dfrac{dD}{dt}=\dfrac{625(3)}{125}=15$

B 16. $\theta=$ angle between $\overline{CD}$ and $\overline{AB}$. $\tan\theta=\dfrac{10t+15t}{100}=\dfrac{t}{4} \Rightarrow \sec^2\theta\dfrac{d\theta}{dt}=\dfrac{1}{4}$
At $t=3 \Rightarrow \theta=\tan^{-1}\left(\dfrac{3}{4}\right) \Rightarrow \sec^2\theta=\dfrac{25}{16} \Rightarrow \dfrac{d\theta}{dt}=\dfrac{4}{25}$

B 17. $y=x^3-3x^2-x+12 \Rightarrow y'=3x^2-6x-1=\dfrac{f(2)-f(1)}{2-1}=-3 \Rightarrow 3x^2-6x+2=0$
$\Rightarrow x=\dfrac{6\pm\sqrt{12}}{6}=1\pm\dfrac{\sqrt{3}}{3} \Rightarrow 1+\dfrac{\sqrt{3}}{3}$ is in the range specified

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C 18. $\int_{-3}^{3}\left(x^3+x^2\right)dx=\int_{-3}^{3}x^2dx=18$

B 19. $s=\int_{0}^{\frac{\pi}{4}}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_{0}^{\frac{\pi}{4}}\sqrt{1+\tan^2 x}\,dx=\int_{0}^{\frac{\pi}{4}}(\sec x)dx=\ln\left(\sqrt{2}+1\right)$

D 20. $r=a+b\cos\theta\Rightarrow$ cardiod. $|a|<|b|\Rightarrow$ loop

B 21. $xy^2+\sin x+y=\frac{x}{y^2}+1\Rightarrow y^2+2xy\frac{dy}{dx}+\cos x+\frac{dy}{dx}=\frac{y^2-2xy\frac{dy}{dx}}{y^4}\Rightarrow 1+1+\frac{dy}{dx}=1\Rightarrow\frac{dy}{dx}=-1$

C 22. $\int_{0}^{0.5}\sqrt{1-x^2}\,dx$ is a piece of a circle $=\frac{\pi}{4}-\left(\frac{\pi}{6}-\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)\right)=\frac{\pi}{12}+\frac{\sqrt{3}}{8}$

D 23. $f(x)=2x^3-6x-34\Rightarrow f(3)=2\Rightarrow g'(2)=\frac{1}{f'(3)}=\frac{1}{48}$

B 24. $V=\int_{0}^{2}\pi x^2dy=\int_{0}^{2}\pi y^4dy=\frac{32\pi}{5}$

C 25. $\sum_{x=1}^{n}\left(\frac{x^2+2n^2}{n^3}\right)=\int_{0}^{1}\left(x^2+2\right)dx=\frac{7}{3}$

A 26. $A=\int_{0}^{\frac{2\pi}{3}}(1+2\cos\theta)^2d\theta-\int_{\frac{2\pi}{3}}^{\pi}(1+2\cos\theta)^2d\theta=\int_{0}^{\frac{2\pi}{3}}(3+4\cos\theta+2\cos2\theta)d\theta-\int_{\frac{2\pi}{3}}^{\pi}(3+4\cos\theta+2\cos2\theta)d\theta$

$=(3\theta+4\sin\theta+\sin2\theta)_{0}^{\frac{2\pi}{3}}-(3\theta+4\sin\theta+\sin2\theta)_{\frac{2\pi}{3}}^{\pi}=\left(2\pi+\frac{3\sqrt{3}}{2}\right)-\left(\pi-\frac{3\sqrt{3}}{2}\right)=\pi+3\sqrt{3}$

D 27. Let h be the height of the cone created. $\Rightarrow V=\frac{\pi r^2h}{3}=\frac{\pi h}{3}\left(36-h^2\right)\Rightarrow\frac{dV}{dh}=\frac{\pi}{3}\left(36-3h^2\right)=0$

$\Rightarrow h=2\sqrt{3}\Rightarrow$ area $=\frac{2\pi\sqrt{36-\left(2\sqrt{3}\right)^2}}{2\pi(6)}(36\pi)=12\pi\sqrt{6}$

D 28. $f^{(a)}(x)=xe^x+(a+1)e^x\Rightarrow f^{(1999)}(x)=xe^x+2000e^x\Rightarrow f^{(1999)}(0)=2000$

B 29. $a_{x1}=0;a_{y1}=-32;a_{x2}=0;a_{y2}=-32;$

$v_{x1}=20\cos(45);v_{y1}=-32t+20\sin(45);v_{x2}=20\cos(30);v_{y2}=-32(t-5)+20\sin(30);$

$s_{x1}=10\sqrt{2}t;s_{y1}=-16t^2+10\sqrt{2}t+640;s_{x2}=10\sqrt{3}t;s_{y2}=-16(t-5)^2+10(t-5)+640;$

Maximum elevation of orange #2 $\Rightarrow v_{y2}=0\Rightarrow t=\frac{85}{16}$

Difference between two oranges: $v_{xd}=10\left(\sqrt{2}-1\right);v_{yd}=10\left(\sqrt{2}-1\right)-160$

$D^2=s_{xd}^2+s_{yd}^2\Rightarrow D\frac{dD}{dt}=s_{xd}\frac{ds_{xd}}{dt}+s_{yd}\frac{ds_{yd}}{dt}=s_{xd}v_{xd}+s_{yd}v_{yd}$

$s_{xd}=\frac{425}{8}\left(\sqrt{2}-\sqrt{3}\right);s_{yd}=\frac{85}{8}\left(5\sqrt{2}-85\right)+450;D=\sqrt{s_{xd}^2+s_{yd}^2};\frac{dD}{dt}\approx155.52$

A 30. $V=\int_{0}^{5}\left(\pi\sqrt{\frac{z}{9}}\sqrt{\frac{z}{16}}\right)dz=\int_{0}^{5}\frac{\pi z\,dz}{12}=\frac{25\pi}{24}$