CALCULUS TEAM KEY JANUARY 8, 1994

1. $-\pi$

2. 3

3. 41/4

4. $-1/(2e^{x} - 1)$

5. 500

6. $(1/e, \infty)$

7. $2\sqrt[3]{4}$

8. $2ex + y = 0$

9. $\sqrt{3}/2$

10. $(\sqrt{6}/3, -20/9)$, $(-\sqrt{6}/3, -20/9)$

11. -5 $g'(0)$ does not exist

12. 125

13. $(-\infty, (-5 - \sqrt{7})/6) \cup ((-5 + \sqrt{7})/6, \infty)$

14. $4x + 2\pi y - 3\pi = 0$

15. -1/24

Calculus Team Solutions

1) $y = \cos \pi x \sin^2 \pi x$

$y = \cos \pi x (\sin \pi x)^2$

$y' = \cos \pi x \, [(2 \sin \pi x)(\cos \pi x)\pi] + [\sin \pi x]^2 [-\sin \pi x]\pi$

at $x = \frac{1}{2}$, $y' = \cos\frac{\pi}{2}\left[\left(2 \sin\frac{\pi}{2}\cos\frac{\pi}{2}\right)\pi\right] + \left(\sin\frac{\pi}{2}\right)^2\left(-\sin\frac{\pi}{2}\right)\pi$

$= 0 + 1(-1)\pi = -\pi$

2) $4e^{x^2y} - 6x + 2y = 0$

$4e^{x^2y}(x^2y' + y \cdot 2x) - 6 + 2y' = 0$

$4e^{x^2y}x^2y' + 2xy(4e^{x^2y}) - 6 + 2y' = 0$

$y' = \left(\dfrac{6 - 8xye^{x^2y}}{4e^{x^2y}x^2 + 2}\right)$

$\lim_{x \to 0} y' = \frac{6}{2} = 3$

3) $A = \lim_{x \to 0^-} \dfrac{|x|}{2x} = -\frac{1}{2}$

$B = \lim_{x \to 0} \dfrac{x^3 + x^2 - 20x}{x^3 - x^2 - 2x} = \lim_{x \to 0} \dfrac{x^2 + x - 20}{x^2 - x - 2} = 10$

$C = \lim_{x \to 0} \dfrac{-\tan x}{x \sec x} = \lim_{x \to 0} \dfrac{-\frac{\sin x}{\cos x}}{\frac{x}{\cos x}} = -1$

$D = \lim_{x \to 0^+} 2\,[\![2x+1]\!] = 2 \lim_{x \to 0^+} [\![2x+1]\!] = 2$

$\therefore (A - 2B)/CD = 41/4$

4) $y' = \dfrac{1}{\frac{2e^x}{\cancel{(2e^x-1)}}} \cdot \dfrac{(2e^x - 1)2e^x - 2e^x(2e^x)}{(2e^x - 1)^{\cancel{2}}}$

$= \dfrac{\cancel{2e^x}\left[(2e^x - 1) - 2e^x\right]}{\cancel{2e^x}(2e^x - 1)} = \dfrac{-1}{2e^x - 1}$

5) $p(x) = 12 - .20\,\dfrac{x - 400}{10}$, $x \geq 400$

$R(x) = x\,p(x) = x\left(12 - .20\,\dfrac{x - 400}{10}\right)$

$= 12x - .20x\left(\dfrac{x - 400}{10}\right)$

$R(x) = 12x - .02x^2 + 8x$ $\Rightarrow 20 - .04x = 0$

$R(x) = 20x - .02x^2$ $\quad .04x = 20$

$R'(x) = 20 - .04x$ $\quad x = 500$

6) $y = x \ln x \Rightarrow y' = x \cdot \frac{1}{x} + \ln x = 1 + l$

$1 + \ln x = 0 \Rightarrow \ln x = -1 \Rightarrow e^{-1} = x$

y': $- - - - - - - \; + + + + + + + +$ (marks at 0, $1/e$, 1)

y'': $- - - - - - - - - \; + + + + + + + + + + +$ (mark at 0)

$y'' = \frac{1}{x} > 0$ when $x > 0$

$\therefore \left(\frac{1}{e}, \infty\right)$ is where y is increasing and concave up.

7) $\frac{dV}{dt} = 4$, $\frac{d(SA)}{dt} = ?$ $\quad \frac{64\pi}{3} = V$

$V = \frac{4}{3}\pi r^3 = \frac{64\pi}{3} \Rightarrow 4r^3 = 64 \Rightarrow r = 2\sqrt[3]{2}$

$\frac{dV}{dt} = 3 \cdot \frac{4}{3}\pi r^2 \frac{dr}{dt} \Rightarrow 4 = 4\pi(2\sqrt[3]{2})^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{4\pi\sqrt[3]{4}}$

$SA = 4\pi r^2$

$\frac{d(SA)}{dt} = 8\pi r \frac{dr}{dt} = \cancel{8}^{2}\cancel{\pi}(2\sqrt[3]{2}) \cdot \frac{1}{\cancel{4}\cancel{\pi}\sqrt[3]{4}}$

$= \dfrac{4\sqrt[3]{2}}{\sqrt[3]{4}} = 4\sqrt[3]{\frac{2}{4}} = 4\sqrt[3]{\frac{1}{2}}$

$= \dfrac{4\sqrt[3]{4}}{2} = 2\sqrt[3]{4}$

8) $y = e^{-2x}$ tangent line through $(0,0)$

$y' = -2e^{-2x}$

$m = \dfrac{e^{-2x} - 0}{x - 0}$

$-2e^{-2x} = \dfrac{e^{-2x}}{x} \Longrightarrow x = -\frac{1}{2}$

If $x = -\frac{1}{2}$ then $y = e^{-2(-1/2)} = e$

Point is $(-1/2, e)$, so

$y - e = -2e(x - (-1/2))$

$y - e = -2e(x + 1/2)$

$y - e = -2ex - e$

$2ex + y = 0$

9) $f(x) = x + \frac{1}{x}$

$f'(c) = 1 - \frac{1}{c^2}$

By MVT $1 - \frac{1}{c^2} = \dfrac{\left(\frac{3}{2} + \frac{1}{3/2}\right) - \left(\frac{1}{2} + \frac{1}{1/2}\right)}{\frac{3}{2} - \frac{1}{2}}$

$1 - \frac{1}{c^2} = -\frac{1}{3}$

$c^2 = 3/4 \Rightarrow c = \pm\frac{\sqrt{3}}{2}$

Choose $\frac{\sqrt{3}}{2}$

10) $y = x^4 - 4x^2$

Inflection Pts. Check y''.

$y' = 4x^3 - 8x$

$y'' = 12x^2 - 8 \Longrightarrow 12x^2 - 8 = 0$

$x^2 = 2/3 \Longrightarrow x = \pm\sqrt{6}/3$

Thus $\left(\frac{\sqrt{6}}{3}, -\frac{20}{9}\right)$ and $\left(-\frac{\sqrt{6}}{3}, -\frac{20}{9}\right)$

11) $g'(-1) = \dfrac{2}{1 + (-1)^2} = 1$ $\varnothing$

$g'(0) = e^0(-\sin 0) + e^0\cos 0 = 1$

$g'(e^2) = 1 + 2[\ln e^2 + 1] = 1 + 2(3) = 7$

$\therefore\ 1 + 1 - 7 = -5$ $g'(0)$ does not exist

12) $f(x) = \dfrac{64}{\sin x} + \dfrac{27}{\cos x}$ on $(0, \pi/2)$

$f'(x) = 64(-\csc x \cot x) + 27(\sec x \tan x)$

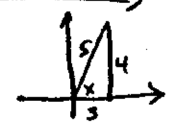
$\dfrac{-64\cos x}{\sin^2 x} + \dfrac{27\sin x}{\cos^2 x} = 0 \Longrightarrow -64\cos^3 x + 27\sin^3 x = 0$

$\tan^3 x = 64/27 \Longrightarrow \tan x = \frac{4}{3}$

$- - - - - - \quad + + + + + +$

$0 \qquad x = \tan^{-1}\left(\frac{4}{3}\right) \qquad \pi/2$

no max, but a min at $x = \tan^{-1}(4/3)$



$f(\tan^{-1}(4/3)) = 64\left(\frac{5}{4}\right) + 27\left(\frac{5}{3}\right) = 125$

13) $f(x) = -4x^3 - 10x^2 - 6x + 4$

$f'(x) = -12x^2 - 20x - 6$

$-12x^2 - 20x - 6 = 0 \Longrightarrow x = \dfrac{-5 \pm \sqrt{7}}{-6}$

$- - - - - - \quad + + + + + + \quad - - - - -$

$\dfrac{-5-\sqrt{7}}{6} \qquad \dfrac{-5+\sqrt{7}}{6}$

$\left(-\infty, \dfrac{-5-\sqrt{7}}{6}\right) \cup \left(\dfrac{-5+\sqrt{7}}{6}, \infty\right)$

14) $y = (\tan x)^x$

$\ln y = x\ln(\tan x)$

$y' = (\tan x)^x\left[\dfrac{x\sec^2 x}{\tan x} + \ln(\tan x)\right]$

At y' at $x = \pi/4$, $y' = -\frac{2}{\pi}$ and $y = \left(\tan\frac{\pi}{4}\right)^{\pi/4} = 1$

$\left(\frac{\pi}{4}, 1\right)$

$y - 1 = -\frac{2}{\pi}\left(x - \frac{\pi}{4}\right)$

$4x + 2\pi y - 3\pi = 0$

15) $\displaystyle\lim_{x\to 0} \frac{\tan^{-1}x - x}{8x^3} = \lim_{x\to 0} \frac{\frac{1}{1+x^2} - 1}{24x^2}$

$\displaystyle = \lim_{x\to 0} \frac{1 - 1 - x^2}{24x^2} = \lim_{x\to 0} \frac{-x^2}{24x^2} = -\frac{1}{24}$