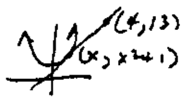
Calculus Team Questions - Solutions   Regional January 12, 1991

1. $f'(x) = 3ax^2 + 2bx + 1$
$f''(x) = 6ax + 2b$
$0 = 6ax + 2b$ when $x = 1/2$
$0 = 3a + 2b$

$f(\frac{1}{2}) = \frac{1}{8}a + \frac{1}{4}b + \frac{1}{2} - 2 = -2$
$a + 2b + 4 - 16 = -16$
$a + 2b = -4$
$-3a - 2b = 0$
$-2a = -4$
$a = 2$
$6 + 2b = 0 \quad 2b = -6$
$b = -3$

2. $\lim\limits_{x \to 0} \dfrac{\tan^2 2x}{x^3 - 2x^2} = \lim\limits_{x \to 0} \dfrac{4 \tan 2x \sec^2 2x}{3x^2 - 4x} = \lim\limits_{x \to 0} \dfrac{(\cancel{4\tan 2x})(2\sec 2x)(2) + 8\sec^4 2x}{6x - 4}$
$= 8/-4 = -2$

3. 

$2x = \dfrac{x^2+1-13}{x-4}$

$2x^2 - 8x = x^2 - 12$
$x^2 - 8x + 12 = 0$
$(x-6)(x-2) = 0$ $\binom{-72}{+37}$

$(6,37)$ and $(2,5)$ are pts of tangency
$f'(6) = 12$ and $f'(2) = 4$
$y - 37 = 12(x-6) \Rightarrow b = -35$
$y - 5 = 4(x-2) \Rightarrow b = -3$

Sum of y intercepts is $-35 + -3 = -38$

4. $f'(x) = (2x+1)^3 \cdot 4(x^3+2)^3 \cdot 3x^2 + (x^3+2)^4 \cdot 3(2x+1)^2 \cdot 2$
$f'(-1) = -1 \cdot 4 \cdot 3 + 1 \cdot 3 \cdot 2 = -12 + 6 = -6$

5. $y' = -2/x^3$ Slope of tangent line is $-2/2^3 = -\frac{1}{4}$ Slope of normal line is 4.
$(2, 1/4)$ is pt on curve $4x - y \mp \frac{31}{4} = 0$ $16x - 4y \mp 31 = 0$
$8 - 1/4$ $16 - 4 \mp 31 = -19$

6. $g(x) = \ln(\arctan \sqrt{3}\, x)$
$g'(x) = \dfrac{1}{\arctan(\sqrt{3}\, x)} \cdot \dfrac{\sqrt{3}}{1 + 3x^2}$ $g'(-1) = \dfrac{1}{\arctan(-\sqrt{3})} \cdot \dfrac{\sqrt{3}}{4} = -\dfrac{\sqrt{3}}{\frac{\pi}{3} \cdot 4} =$
$-\dfrac{\sqrt{3} \cdot 3}{4\pi} = \dfrac{k}{\pi}$ $\boxed{k = -\dfrac{3\sqrt{3}}{4}}$

7. $y = x^2 e^{-x}$
$y' = -x^2 e^{-x} + 2xe^{-x}$
$y'' = +x^2 e^{-x} + {}^-2xe^{-x} + {}^-2xe^{-x} + 2e^{-x}$
$y'' = x^2 e^{-x} - 4xe^{-x} + 2e^{-x}$
$e^{-x}(x^2 - 4x + 2) = 0$
$x = \dfrac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$ ← Must be x coordinates of inflection since there are 2 of them
$E = 2 + \sqrt{2}$, $F = 2 - \sqrt{2}$

$(-x^2 + 2x)e^{-x} = 0$
$-x^2 + 2x = 0$
$x(2 - x) = 0$
$x = 0$ or $x = 2$

	f'
$(-\infty, 0)$	$-$
$(0, 2)$	$+$
$(2, \infty)$	$-$

$A = 0$
$C = 2$

$E + F + A + C = 6$

8. $x(\cos y) = -\pi \sin(x+y)$

at $x=\pi$; $\pi\cos y = -\pi\sin(\pi+y)$

$\cancel{\pi}\cos y = \cancel{\pi}\sin y$

$1 = \tan y$

$\underline{y = \pi/4}$

$-x\sin y\frac{dy}{dx} + \cos y = -\pi\cos(x+y)\left[1+\frac{dy}{dx}\right]$

$-\pi\sin\frac{\pi}{4}\frac{dy}{dx} + \cos\frac{\pi}{4} = -\pi\cos\left(\frac{3\pi}{4}\right)\left(1+\frac{dy}{dx}\right)$

$-\pi\frac{\sqrt{2}}{2}\frac{dy}{dx} + \frac{\sqrt{2}}{2} = -\pi\cdot -\frac{\sqrt{2}}{2}\cdot\left(1+\frac{dy}{dx}\right)$

$-\pi\frac{\sqrt{2}}{2}\frac{dy}{dx} - \pi\frac{\sqrt{2}}{2}\frac{dy}{dx} = \pi\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}$

$-\pi\sqrt{2}\frac{dy}{dx} = \frac{\sqrt{2}}{2}(\pi-1)$

$\frac{dy}{dx} = \frac{\cancel{\sqrt{2}}}{2}(\pi-1)\cdot -\frac{1}{\pi\cancel{\sqrt{2}}} = \boxed{\frac{1-\pi}{2}}$

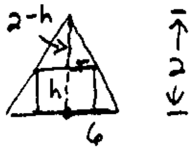
9. $e^{xy} - 4y = 0$

if $x=0$, $y = 1/4$

$e^{xy}\left(x\frac{dy}{dx}+y\right) - 4\frac{dy}{dx} = 0$

$(xe^{xy}-4)\frac{dy}{dx} = -ye^{xy}$

$\frac{dy}{dx} = \frac{-ye^{xy}}{xe^{xy}-4}$ $\qquad \lim_{x\to 0}\frac{-ye^{xy}}{xe^{xy}-4} = \frac{-1/4}{-4} = \boxed{1/16}$

10. 

$V = \pi r^2 h$ $\qquad$ But $\frac{2-h}{r} = \frac{\cancel{2}}{\cancel{6}}\frac{1}{3}$ $\qquad r = 6-3h$

$r^2 = 36-36h+9h^2$

$V = \pi(36h - 36h^2 + 9h^3)$

$V' = \pi(36 - 72h + 27h^2)$ $\qquad V' = 0 \Rightarrow 3h^2 - 8h + 4 = 0$

$(3h-2)(h-\cancel{2}) = 0$

$h = 2/3$

11. $A = \lim_{x\to 1}\frac{\cancel{\sqrt{x}-1}\ -1}{\cancel{(1-\sqrt{x})}(1+\sqrt{x})} = -\frac{1}{2}$

$B = f'(1/2) = \frac{1}{\sqrt{1-1/4}} = \frac{1}{1}\cdot\frac{2}{\sqrt{3}} = 2/\sqrt{3}$ $\qquad C = \left(+\frac{1}{2\sqrt{1}}\right)^{-1} = 2$

$\left[(\text{Arc sec } B)^C\right]^A = \left[(\pi/6)^2\right]^{-1/2} = (\pi/6)^{-1} = 6/\pi$

12. $y = e^x\cos x \Rightarrow y' = -e^x\sin x + e^x\cos x \Rightarrow y'' = \cancel{-e^x\cos x} + -e^x\sin x + -e^x\sin x + \cancel{e^x\cos x}$

$= -2e^x\sin x$

$-2\frac{dy}{dx} = +2e^x\sin x \mp 2e^x\cos x$

$\frac{d^2y}{dx^2} + -2\frac{dy}{dx} + ky = -2e^x\cancel{\sin x} + 2e^x\cancel{\sin x} - 2e^x\cos x + Ke^x\cos x = 0$ $\qquad \boxed{k=+2}$

13. $V_{AVE} = \frac{\tan\pi/3 - \tan\pi/6}{\frac{2\pi}{3} - \pi/3} = \frac{\sqrt{3} - \frac{1}{\sqrt{3}}}{\pi/3} = \frac{3-1}{\frac{\pi\cdot\sqrt{3}}{3\sqrt{3}}} = \boxed{\frac{2\sqrt{3}}{\pi}}$

14. $g(-\ln 4) = g(\ln\frac{1}{4})$ $\qquad$ ($f(g(x)) = x$ since f & g are inverse function)

$x = \ln\frac{y+1}{y}$ $\qquad \frac{y+1}{y} = e^x$ $\quad y - ye^x = 1$ $\quad y = \frac{-1}{1-e^x}$ $\qquad g(x) = \frac{-1}{1-e^x} \Rightarrow g(\ln\frac{1}{4}) =$

$\frac{-1}{1-1/4} = -4/3$

15. $f(x) = \ln(1+\sin x)$

$f'(x) = \frac{\cos x}{1+\sin x}$ $\qquad f'(\pi/6) = \frac{\sqrt{3}/2}{1+1/2} = \frac{\sqrt{3}}{2+1} = \sqrt{3}/3$