Mu Alpha Theta National Convention 2004
Theta State Bowl Answers

#			#	
1-1	2^{x+2}		6-11	$\frac{1203}{370}$
1-2	4x		6-12	25
2-3	$\sqrt{3}$		7-13	7700π
2-4	17		7-14	$\frac{\sqrt{3}-1}{2}$
3-5	$\frac{3\sqrt{3}}{2}$		8-15	0
3-6	$\frac{\pi}{3}-\frac{\sqrt{3}}{4}$		8-16	$12\frac{3}{8}$ kg
4-7	$4\sqrt{6}$		9-17	12
4-8	$\frac{8}{5}$		9-18	$y=\frac{4}{3}x-\frac{13}{3}$ and $y=\frac{-4}{3}x-\frac{5}{3}$
5-9	$\frac{1}{2}$		10-19	-1
5-10	$(x-2)^2+(y+4)^2=6$		10-20	35

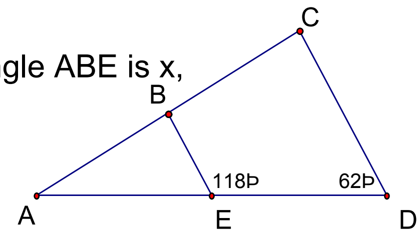
Round 1

1. Simplify for all x: $2^x + 2^x + 2^x + 2^x$.

   Answer: 2^{x+2}

   Solution: $4\left(2^x\right) = 2^2\left(2^x\right) = 2^{x+2}$

2. In the given triangle, AB = BC. If the area of triangle ABE is x, what is the area of triangle ACD?



   Answer: 4x

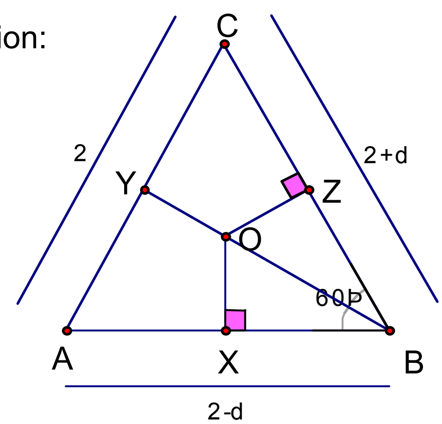
Solution: Since $118^\circ + 62^\circ = 180^\circ$, $\overline{BE} \parallel \overline{CD}$. Triangle ACD and triangle ABE are similar with sides in ratio of 2:1. Their areas are in ratio of 4:1. Area of triangle ACD = 4x

Round 2

3. The sides of a triangle are in an arithmetic progression with the middle term = 2 and the angle opposite
   the side of 2 is 60°. Find the area of the triangle.

   Answer: $\sqrt{3}$

   Solution:



$(2-d)+(2)+(2+d)=$ the perimeter = 6 the semiperimeter = 3

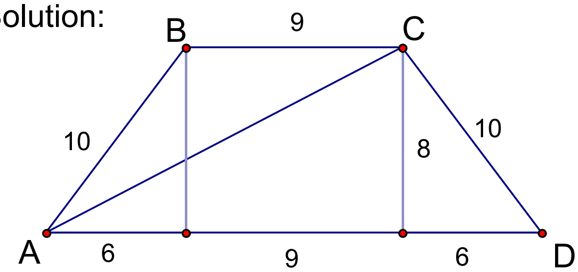
$CY+AY=2$ $CZ+AX=2$ $AB+BC=4$ $BZ=BX$ $\therefore$ both = 1

$\text{AngleOBX}=30^\circ$ triangleOB is a 30, 60, 90 triangle OX = the inradius = $\frac{\sqrt{3}}{3}$

$\text{Area}=\text{inradius(semiperimeter)}=\left(\frac{\sqrt{3}}{3}\right)(3)=\sqrt{3}$

4. The perimeter of trapezoid ABCD is 50. If the two bases BC = 9 and AD = 21, what is the length of
the diagonal AC?

Answer: 17
Solution:



Since the perimeter is 50 and the legs are congruent, the legs = $\frac{50-(21+9)}{2}=10$

Dropping $\perp$'s from B and C forms rt triangles 6, 8, & 10. The diagonal forms a rt triangle of 8, 15, & hypotenuse.

$AC=\sqrt{(15)^2+(8)^2}=17$

Round 3

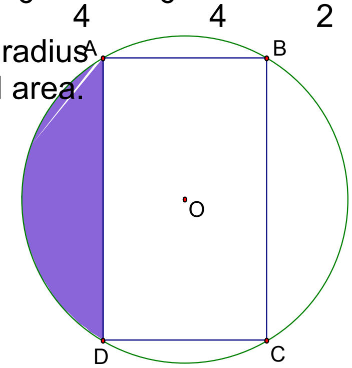
5. A square and an hexagon have the same perimeter. If the area of the square is 2.25, what is the area of
the hexagon?

Answer: $\frac{3\sqrt{3}}{2}$

Solution:

Area of square = 2.25 means side of square = 1.5. Perimeter of square = 6 $\Rightarrow$ Perimeter of hexagon = 6 $\Rightarrow$ side of hexagon = 1 $\Rightarrow$ Area = $6\frac{s^2\sqrt{3}}{4}=6\frac{1^2\sqrt{3}}{4}=\frac{3\sqrt{3}}{2}$

6. In the figure, rectangle ABCD is inscribed in a circle. If the radius of the circle is 1 and AB = 1, what is the area of the shaded area.



Answer: $\frac{\pi}{3}-\frac{\sqrt{3}}{4}$

Solution:

Triangle AOB is equilateral $\Rightarrow$ m$\angle$AOD = 120°

Area of sector = $\frac{1}{3}\pi r^2=\frac{1}{3}\pi(1)^2=\frac{\pi}{3}$

Area of segment = sector - triangle =

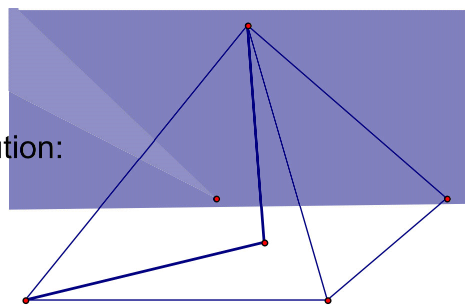
$\frac{\pi}{3}-\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\sqrt{3}\right)=\frac{\pi}{3}-\frac{\sqrt{3}}{4}$

Round 4

7. A regular pyramid is composed of a square base of area 12 and four equilateral triangles. What is the
volume of the pyramid?

Answer: $4\sqrt{6}$

Solution:



Square base with area = 12 $\Rightarrow$ side = $2\sqrt{3}$

The right triangle with hypotenuse = a lateral edge of the pyramid = $2\sqrt{3}$, one side = $\frac{1}{2}$ diagonal of square = $\sqrt{6}$ and the 3rd side is the height. $\left(2\sqrt{3}\right)^2=h^2+\left(\sqrt{6}\right)^2$

$h=\sqrt{6}\quad\Rightarrow\quad V=\frac{1}{3}Bh=\frac{1}{3}(12)\sqrt{6}=4\sqrt{6}$

8. What is the sum of the infinite geometric series $2+\left(-\frac{1}{2}\right)+\left(\frac{1}{8}\right)+\left(-\frac{1}{32}\right)+\ldots$

Answer: $\frac{8}{5}=1\frac{3}{5}$

Solution:

The first term $a_1=2$, and the common ratio $r=\frac{-\frac{1}{2}}{2}=-\frac{1}{4}\quad\Rightarrow$ Sum = $\frac{2}{1-\left(\frac{-1}{4}\right)}=\frac{2}{\frac{5}{4}}=\frac{8}{5}$

Round 5

9. If $4^{2x+2}=64$, then $x=$

Answer: $x=\frac{1}{2}$

Solution: $4^{2x}\left(4^2\right)=64 \quad \Rightarrow \quad 4^{2x}=\frac{64}{16}=4 \quad \Rightarrow \quad 2x=1 \quad \Rightarrow \quad x=\frac{1}{2}$

10. Consider the ellipse with the equation $9x^2+4y^2-36x+32y+64=0$. Find the equation of the circle, in graphing form, with its center at the center of the ellipse and its area the same as the ellipse.

Answer: $(x-2)^2+(y+4)^2=6$

The ellipse has the equation $\frac{(x-2)^2}{4}+\frac{(y+4)^2}{9}=1 \quad \Rightarrow$ center at $(2, -4)$

Solution: Area of ellipse = $\pi ab=(2)(3)\pi=6\pi \quad \Rightarrow \quad \pi r^2=6\pi \quad \Rightarrow \quad r=\sqrt{6}$

For the circle $(x-2)^2+(y+4)^2=6$

Round 6

11. Change 3.2513513513513… to a fraction. At lowest terms

Answer : $\frac{1203}{370}$

Solution:

Let N = 3.2513513513...

10000N = 32513.513513...

-10N = -32.513513513.....

9990N = 32481 $\Rightarrow$ N = $\frac{32481}{9990}=\frac{1203}{370}$

12. Solve over the real numbers: $x-\sqrt{x}-20=0$

Answer: x=25

Solution:

Factoring $x^{\frac{2}{2}}-x^{\frac{1}{2}}-20=0 \quad \Rightarrow \quad \left(x^{\frac{1}{2}}-5\right)\left(x^{\frac{1}{2}}+4\right)=0 \quad \Rightarrow$

$x^{\frac{1}{2}}=5 \quad \Rightarrow \quad x=25 \qquad x^{\frac{1}{2}}=-4$ impossible

Round 7

13. A goat is tethered by a 100 ft. rope attached to an outside corner of an 80 ft. by 80 ft. square barn.

How much grazing area outside the barn can the goat reach?

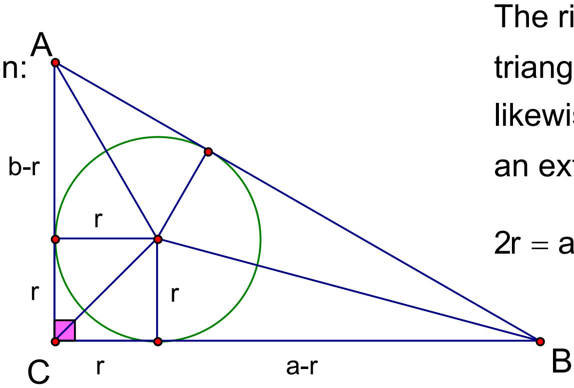
Answer: 7700π sq units

Solution: The grazing area consists of $\frac{3}{4}$ of a circle with radius 100 and $2\left(\frac{1}{4}\text{ of circle of radius }20\right) \Rightarrow A = \frac{3}{4}(\pi)(10000) + \frac{1}{2}(\pi)(400) = 7700$

14. A triangle has sides of lengths 1, 2, and $\sqrt{3}$. Find the radius of a circle inscribed in the triangle.

Answer: $\frac{\sqrt{3}-1}{2}$

Solution:



Drop $\perp$'s from the incenter to the 3 sides and draw segments from the incenter to the vertices. The right angle is bisected and creates isosceles triangles. Side a is divided into r and $a-r$, a and likewise for side b. $\therefore$ since they are tangents from an external point, $(a-r)+(b-r)=c \Rightarrow$

$$2r = a+b-c \Rightarrow r = \frac{1+\sqrt{3}-2}{2} = \frac{\sqrt{3}-1}{2}$$

Round 8

15. Evaluate $\log\frac{7}{8} + \log\frac{9}{14} - 2\log\frac{3}{4}$ where logs are to base 2.

Answer: 0

Solution: $\log\frac{\left(\frac{7}{8}\right)\left(\frac{9}{14}\right)}{\left(\frac{3}{4}\right)^2} = \log\frac{\left(\frac{9}{16}\right)}{\left(\frac{9}{16}\right)} = \log 1 = 0$

16. A snow man is made using three balls of snow with diameters of 20 cm, 30 cm, and 40 cm. If the
head of the snow man weighs 1 kg, what is the total weight of the snow man? (the head is the 20 cm
snowball)

Answer: $12\frac{3}{8}$ kg

Weight is proportional to volume, and volume is proportional to the cube of the diameter.

Solution: weight of 30 cm ball = x $\quad \dfrac{(20)^3}{(30)^3} = \dfrac{1}{x} \quad \Rightarrow \quad x = \dfrac{(30)^3}{(20)^3}(1) = \dfrac{27}{8}$

weight of 40 cm ball = $\dfrac{(40)^3}{(20)^3}(1) = 8 \quad \Rightarrow \quad$ Total = $1 + \dfrac{27}{8} + 8 = 12\dfrac{3}{8}$

Round 9

17. Find the positive integer x for which $\dfrac{2^{x^2}}{4^{5x}} = 8^8$.

Answer: 12

Solution: $\dfrac{2^{x^2}}{(2^2)^{5x}} = (2^3)^8 \quad \Rightarrow \quad x^2 - 10x = 24 \quad \Rightarrow \quad x^2 - 10x - 24 = 0 \quad \Rightarrow$

$(x-12)(x+2) = 0 \quad \Rightarrow \quad x = 12$

18. Find the equations of the asymptotes for $16x^2 - 9y^2 - 32x - 54y - 209 = 0$ in slope intercept form.

Answer: $y = \dfrac{4}{3}x - \dfrac{13}{3}$ and $y = -\dfrac{4}{3}x - \dfrac{5}{3}$

Solution: $\dfrac{(x-1)^2}{9} - \dfrac{(y+3)^2}{16} = 1 \quad \therefore$ The asymptotes are $y+3 = \pm(x-1)$

In slope intercept form $y = \dfrac{4}{3}x - \dfrac{13}{3}$ and $y = -\dfrac{4}{3}x - \dfrac{5}{3}$

Round 10

19. Simplify: $i^{447763898275634058776 34}$

Answer: -1

$i^1 = i \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1$ After 4 it repeats

Solution: You only need to divide the last two digits of the exponents by 4 $\Rightarrow$

$\frac{34}{4} = 8$ with remainder = 2 $\Rightarrow$ value = -1

20. In an arithmetic progression of positive numbers, the common difference is three times the first term,
and the sum of the first five terms is equal to the square of the first term. Find the first term.

Answer: 35

Solution: $d = 3a \quad \Rightarrow \quad a + 4a + 7a + 10a + 13a = a^2 \quad \Rightarrow \quad a^2 = 35a \quad \Rightarrow \quad a = 35$