1992 Mu Alpha Theta National Convention

Sequences and Series Answers

1. D
2. E
3. A
4. B
5. C
6. D
7. B
8. C
9. C
10. B
11. E
12. E
13. B
14. B
15. B
16. C
17. C
18. E
19. C
20. E
21. D
22. C
23. D
24. D
25. A
26. D
27. A
28. C
29. D
30. B

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1) $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}\cdots = \frac{1}{1-(-\frac{1}{3})} = \frac{1}{4/3} = \boxed{\frac{3}{4}}$

2) $\frac{1}{7} = 0.\overline{142857}$ $\therefore$ $22 \equiv 4 \pmod{6}$ $\therefore$ THE 22^{nd} DIGIT IS THE SAME AS THE 4^{TH}, $\boxed{8}$.

3) $1-2+2-4+\cdots+15-30 = -1-2-3-\cdots-15 = -1(1+2+3+\cdots+15) = -1\left(\frac{15(16)}{2}\right) =$
$= \boxed{-120}$

4) $ar = 4$ $\therefore$ $ar^5 = 16$ $\therefore$ $(ar)(ar^5) = (ar^3)^2 = 64$ $\therefore$ $ar^3 = \pm 8$

SINCE THE 4^{th} TERM IS ar^3, AND BOTH ar & r^2 ARE POSITIVE, $\boxed{8}$ IS OUR FOURTH TERM.

5) $d-a = 3(b-a)$ $\therefore$ $b-a = r/3$ $\therefore$ $c-a = 2(b-a) = \boxed{\frac{2r}{3}}$

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(6) $(-i)^{15} = -i^{15} = -i^3 = -(-i) = \boxed{i}$

(7) $\underline{50} - 10 = \underline{40} - 8 = \underline{32} - 6 = \underline{26} - 4 = \underline{22} - 2 = \underline{20} - 0 = \boxed{20}$

(8) $a = 1 \therefore a + 6r = 17 \therefore a + 3r = \dfrac{a + (a+6r)}{2} = \dfrac{1+17}{2} = \boxed{9}$

(9) AT THE END OF x HOURS, BEFORE ACCELERATING, THE MAN IS TRAVELLING $5x$ MPH; THUS AFTER x HOURS, HE HAS TRAVELLED $5 + 10 + \cdots + 5x = \frac{x}{2}(5 + 5x)$ MILES. TO FIND THE GREATEST NUMBER OF WHOLE HOURS HE CAN TRAVEL AND NOT COVER 250 MILES, WE SOLVE FOR THE GREATEST INTEGER x SUCH THAT $\frac{x}{2}(5+5x) \leq 250 \therefore$ $x + x^2 \leq 100 \therefore x = 9$ IS OUR MAXIMUM INTEGER. AFTER 9 HOURS, THE MAN HAS TRAVELLED $\frac{9}{2}(50) = 225$ MILES, AND THEN THE MAN ACCELERATES TO 50 MPH. IT TAKES $\frac{1}{2}$ hr TO TRAVEL THE REMAINING 25 mi AT 50 MPH. THUS, OUR ANSWER IS $\boxed{9\frac{1}{2} \text{ HOURS}}$

(10) $\displaystyle\prod_{i=1}^{20} \left(\frac{i}{i+1}\right) = \left(\frac{1}{\cancel{2}}\right)\left(\frac{\cancel{2}}{\cancel{3}}\right)\left(\frac{\cancel{3}}{\cancel{4}}\right)\cdots\left(\frac{\cancel{19}}{\cancel{20}}\right)\left(\frac{\cancel{20}}{21}\right) = \boxed{\frac{1}{21}}$

SEQUENCES & SERIES

(11) CONSIDER THE STANDARD INFINITE GEOMETRIC SERIES:

$1+y+y^2+\cdots = \frac{1}{1-y}$ $\therefore$ LET $y=(-2x^2)$ $\therefore$

$$\frac{1}{1-(-2x^2)} = \frac{1}{1+2x^2} = 1+(-2x^2)+(-2x^2)^2+(-2x^2)^3+\cdots = 1-2x^2+4x^4-8x^6+16x^8+\cdots$$

(12) $$\sum_{n=1}^{\infty}\left(\frac{n}{(n+1)(n+2)}\right) = \sum_{n=1}^{\infty}\left(\frac{2}{n+2}-\frac{1}{n+1}\right) = 2\left(\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\cdots\right)-\left(\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\cdots\right) =$$

$$= \left(\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}+\cdots\right)-\tfrac{1}{2}$$

THIS DIVERGES. PROOF: $\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\cdots+\frac{1}{16}\right)+\cdots >$

$> 2\left(\frac{1}{4}\right)+4\left(\frac{1}{8}\right)+8\left(\frac{1}{16}\right)+\cdots = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$ WHICH OBVIOUSLY IS NOT FINITE.

(13) $$X = \cfrac{2}{2+\cfrac{4}{2+\cfrac{4}{2+\cdots}}} = \frac{2}{2+2x} \therefore X = \frac{1}{1+x} \therefore x^2+x-1=0 \therefore X = \frac{-1\pm\sqrt{5}}{2}$$

SINCE $x>0$, $\boxed{X=\frac{\sqrt{5}-1}{2}}$.

(14) REWRITE THE SEQUENCE: $\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \frac{6}{64}, \cdots \frac{n}{2^n}$

$$\therefore \frac{a}{b} = \frac{5/32}{6/64} = \frac{10}{6} = \boxed{\frac{5}{3}}$$

(15) $\ln b - \ln a = 1 \therefore \ln\frac{b}{a}=1 \therefore \frac{b}{a}=e$

$\ln c - \ln b = 1 \therefore \ln\frac{c}{b}=1 \therefore \frac{c}{b}=e$ $\therefore$ $\boxed{\text{GEOMETRIC SEQUENCE, RATIO } e}$

Sequences & Series

(16) Since $\sum_{i=0}^{n} \binom{n}{i} = 2^n$, the expression equals $\frac{2^n}{2^m} = 2^{n-m}$, which is an integer if & only if $n-m \geq 0 \therefore \boxed{n \geq m}$

(17) $\sum_{i=1}^{10} \sum_{k=1}^{i} \binom{i}{k} = \sum_{i=1}^{10} \sum_{k=0}^{i} \binom{i}{k} - \sum_{i=1}^{10} \binom{i}{0} = \sum_{i=1}^{10} (2^i) - 10 = 2+4+8+\cdots+2^{10} - 10 =$

$= 2^{11} - 2 - 10 = 2048 - 12 = \boxed{2036}$

(18) $2^1 \equiv 2 \pmod 9 \therefore 2^2 \equiv 4 \pmod 9 \therefore 2^3 \equiv 8 \pmod 9 \therefore 2^4 \equiv 7 \pmod 9 \therefore 2^5 \equiv 5 \pmod 9 \therefore$

$2^6 \equiv 1 \pmod 9 \quad \therefore \quad 71 \equiv 5 \pmod 6 \therefore 2^{71} \equiv (2^{11})^6 2^5 \pmod 9 \equiv 1(5) \pmod 9 \quad \boxed{5}$

(19) $\sum_{i=1}^{25} \left(i(2i-1)\right) = 2\sum_{i=1}^{25} i^2 - \sum_{i=1}^{25} i = 2\left(\frac{25(26)(51)}{6}\right) - \frac{25(26)}{2} =$

$= 25(26)(17) - 25(13) = 33(25)(13) = \boxed{10725}$

(20) Counterexamples:

I) $S = 1-1+1-1+1-1+\cdots$ Infinitely many terms equal to 1, but $0 \leq S \leq 1$

II) $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-1/2} = 2$ All terms positive, but $S = 2$

III) $S = 1-1+1-1+1-1+\cdots$ Infinitely many terms less than zero, but $0 \leq S \leq 1$

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(21) LET THE TERMS BE $a-r, a, a+r$ $\therefore$ $a(a-r)(a+r) = a^3 - r^2 a = 3a$ $\therefore$

SINCE $a \neq 0$ $\therefore$ $a^2 - r^2 = 3$ $\therefore$ $a^2 = r^2 + 3$ $\therefore$ $a = \pm\sqrt{3+r^2}$.

SINCE a IS AN INTEGER (a IS MIDDLE TERM), $3+r^2$ MUST BE A PERFECT SQUARE. THIS IS TRUE FOR $r = 1, \sqrt{6}, \sqrt{13}, \sqrt{22}, \sqrt{33}, \ldots$

THE SUM OF THE SQUARES OF THE 1^{ST} 5 IS $1+6+13+22+33 = \boxed{75}$

(22) ALL TRIANGULAR NUMBERS CAN BE WRITTEN IN THE FORM $\binom{n}{2}, n \geq 2$.

THUS, OUR SUM IS $\sum_{n=2}^{31} \binom{n}{2} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{31}{2} =$

$$= \binom{3}{3} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{31}{2} = \binom{4}{3} + \binom{4}{2} + \binom{5}{2} + \cdots + \binom{31}{2} = \binom{5}{3} + \binom{5}{2} + \cdots + \binom{31}{2} =$$

$$= \binom{31}{3} + \binom{31}{2} = \binom{32}{3} = \frac{32 \cdot 31 \cdot 30}{3 \cdot 2 \cdot 1} = \boxed{4960}$$

(23) $f(2) = F_0 f(0) - F_1 f(1) = F_1 f(0) - F_1 f(1)$ $\therefore$ $f(3) = F_0 f(1) - f(2) =$

$= F_0 f(1) - (F_1 f(0) - F_1 f(1)) = F_2 f(1) - F_1 f(0)$ $\therefore$ GIVEN

$f(2n) = F_{2n-2} f(0) - F_{2n-1} f(1)$ & $f(2n+1) = F_{2n} f(1) - F_{2n-1} f(0)$

WE HAVE: $f(2n+2) = f(2n) - f(2n+1) = (F_{2n-2} + F_{2n-1}) f(0) - (F_{2n-1} + F_{2n}) f(1) =$

$= F_{2n} f(0) - F_{2n+1} f(1)$

$f(2n+3) = f(2n+1) - f(2n+2) = (F_{2n} + F_{2n+1}) f(1) - (F_{2n} + F_{2n-1}) f(0) =$

$= F_{2n+2} f(1) - F_{2n+1} f(0)$.

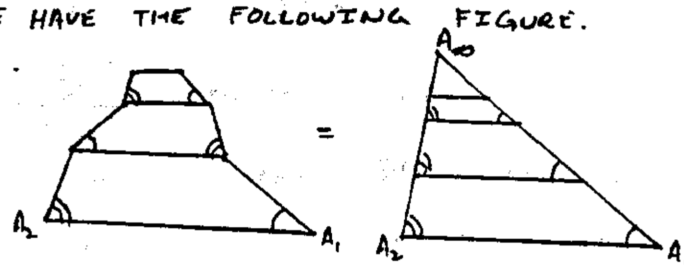
THUS, WE HAVE PROVEN BY MATHEMATICAL INDUCTION: $f(2n) = F_{2n-2} f(0) - F_{2n-1} f(1)$

$\therefore$ $f(100) = F_{98} f(0) - F_{99} f(1) = 3F_{98} + F_{99} = 2F_{98} + F_{98} + F_{99} = \boxed{2F_{98} + F_{100}}$

(24) FROM THE SIMILARITY STATEMENT, WE HAVE THE FOLLOWING FIGURE.

THE RATIO OF SIMILAR SIDES IS $\frac{4}{8} = \frac{1}{2}$.

BY SIMILARITY WE HAVE THE SHOWN EQUAL ANGLES. SUPPOSE WE FLIP EVERY OTHER TRAPEZOID (NOT AFFECTING AREA, AS



LECTIONS DON'T CHANGE AREA). BY THE EQUAL ANGLES MADE BY PARALLEL LINES, WE SEE WE HAVE A TRIANGLE. HENCE, $A_1 A_\infty = 6 + 3 + \frac{3}{2} + \ldots = \frac{6}{1 - \frac{1}{2}} = 12$; $A_2 A_\infty = \frac{5}{1 - \frac{1}{2}} = 10$. $A_1 A_2 = 8$, so OUR AREA IS $\sqrt{(15)(5)(7)(3)} = 15\sqrt{7}$

SEQUENCES & SERIES

25) A HARMONIC SERIES IS A TERM BY TERM RECIPROCAL OF AN ARITHMETIC SERIES. HENCE, $\frac{1}{n} + \frac{1}{n+r} + \frac{1}{n+2r} + \frac{1}{n+3r} + \cdots$ IS A HARMONIC SERIES. IF IT IS ALSO AN ARITHMETIC SERIES, WE MUST HAVE A CONSTANT DIFFERENCE BETWEEN TERMS: $\frac{1}{n+r} - \frac{1}{n} = \frac{1}{n+2r} - \frac{1}{n+r} \therefore$

$n(n+2r) - (n+r)(n+2r) = n(n+r) - n(n+2r) \therefore -r(n+2r) = -rn \therefore -2r^2 = 0 \therefore$

$r = 0$. HENCE, OUR SERIES IS $\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \ldots$ WHICH CANNOT HAVE A FINITE SUM. IT $\boxed{\text{DIVERGES}}$.

26) $$\sum_{i=0}^{45} (44-i)\binom{45}{i} = 44\sum_{i=0}^{45}\binom{45}{i} - \sum_{i=0}^{45} i\binom{45}{i} = 44(2)^{45} - \sum_{i=1}^{45} i\binom{45}{i} - 0\binom{45}{0} =$$

$$= 44(2)^{45} - \sum_{i=1}^{45}\left(\frac{i(45!)}{(45-i)!\,i!}\right) = 44(2)^{45} - \sum_{i=1}^{45}\left(\frac{45(44!)}{(44-(i-1))!\,(i-1)!}\right) =$$

$$= 44(2^{45}) - 45\sum_{i=0}^{44}\left(\frac{44!}{(44-i)!\,i!}\right) = 44(2^{45}) - 45(2^{44}) = \boxed{43(2^{44})}$$

27) $$\sum_{n=0}^{\infty}\left(\frac{\sin(n\theta)}{3^n}\right) = \sum_{n=0}^{\infty}\frac{Im(e^{in\theta})}{3^n} = Im\left(\sum_{n=0}^{\infty}\frac{e^{in\theta}}{3^n}\right) = Im\left(\sum_{n=0}^{\infty}\left(\frac{e^{i\theta}}{3}\right)^n\right) =$$

$$= Im\left(\frac{1}{1-\frac{e^{i\theta}}{3}}\right) = Im\left(\frac{3}{3-\cos\theta - i\sin\theta}\right) = Im\left(\frac{3(3-\cos\theta + i\sin\theta)}{(3-\cos\theta)^2 + \sin^2\theta}\right) =$$

$$= \frac{3\sin\theta}{9-6\cos\theta+\cos^2\theta+\sin^2\theta} = \frac{3\sin\theta}{10-6\cos\theta} = \frac{3(1/3)}{10-6\left(\frac{2\sqrt{2}}{3}\right)} = \frac{1}{10-4\sqrt{2}} = \boxed{\frac{5+2\sqrt{2}}{34}}$$

28) IN A LIST FROM 1 TO n, OMITTING ONE NUMBER, THE MAXIMUM AVERAGE IS ATTAINED BY OMITTING 1, THE MINIMUM BY OMITTING n. THESE AVERAGES ARE $\frac{2+n}{2}$ AND $\frac{1+n-1}{2} = \frac{n}{2}$ RESPECTIVELY. HENCE, $\frac{2+n}{2} \geq \frac{94}{5} \geq \frac{n}{2} \therefore 10+5n \geq 188 \geq 5n \therefore$ THUS, $10+5n \geq 188$ YIELDS $n \geq 35.6$ AND $188 \geq 5n$ YIELDS $n \leq 37.6$. SINCE n IS AN INTEGER, n IS 36 OR 37. TO ATTAIN OUR AVERAGE WE HAVE $\frac{\frac{n(n+1)}{2} - x}{n-1}$, WHERE x IS THE OMITTED NUMBER. THE AVERAGE, 18.8, HAS A FACTOR OF 5 IN THE DENOMINATOR. THUS, $n-1$ IS A MULTIPLE OF 5. HENCE, $n = 36$. SOLVING $\frac{\frac{36(37)}{2} - x}{35} = \frac{94}{5}$ YIELDS $666 - x = 658 \therefore x = \boxed{8}$

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29) THIS IS A STRAIGHTFORWARD CASE OF THE BINOMIAL IDENTITY:

$$\sum_{i=0}^{n} \binom{q+n-i}{n-i}\binom{p+i}{i} = \binom{p+q+n+1}{n} \quad \therefore \; q=12,\; p=15,\; n=10 \;\therefore \boxed{\binom{38}{10}}$$

30) $19^{92} \pmod{1000} \equiv (-1+20)^{92} \pmod{1000} \equiv (-1)^{92} + \binom{92}{1}(-1)^{91}(20) + \binom{92}{2}(-1)^{90}(20)^2.$

ALL OTHER TERMS HAVE FACTORS OF 1000, HENCE ARE CONGRUENT 0 MODULO 1000. THUS, WE HAVE:

$$19^{92} \equiv (1 - 1840 + 1674400) \pmod{1000} \equiv 1 - 840 + 400 \pmod{1000} \equiv -439 \pmod{1000} \equiv$$
$$\equiv 561 \pmod{1000} \quad \therefore \quad 5+6+1 = \boxed{12}$$