1999 Mu Alpha Theta National Convention
Functions Solutions
Open Division

NOTA stands for None of these answers

Unless otherwise stated, the domain will be all real numbers which are meaningful to the problem.

1. Given $f(x) = 2x + 7$ and $g(x) = 3x + n$, find n so that $f(g(x)) = g(f(x))$

Ans: C
Solution: Solve $2(3x + n) + 7 = 3(2x + 7) + n$ for n. $n = 14$

2. If $f(x) = 2^{x+1}$ and $g(x) = 3^x$ find $f(g(1))$

Answer: D
Solution: $g(1) = 3$ $f(3) = 16$

3. Suppose $f(x)$ is an even function and $g(x)$ is an odd function. What kind of function will $h(x) = f(x) \bullet g(x)$ be?

Ans: B
Solution: It is odd because $h(-x) = f(-x) \bullet g(-x) = f(x) \bullet (-g(x)) = -f(x) \bullet g(x) = -h(x)$

4. Find the value of m and b so that $f(x) = \begin{cases} x^2 - 6x + 14 & \text{if } x > 3 \\ mx + b & \text{if } -3 \leq x \leq 3 \\ x^2 + 6x + 8 & \text{if } x < -3 \end{cases}$ is continuous.

Ans: D
Solution: The vertex points of the 2 parabolas are (3,5) and (-3, -1). Using these as the endpoints of the line segment you can find the equation of the line between them to be $y = x + 2$

5. Suppose $f(x)$ is a polynomial with integer coefficients for which 3 and 13 are both roots. Which of the following could possibly be the value of $f(10)$?

Answer: D
Solution: The given information implies that $f(x) = (x - 3)(x - 13)g(x)$ where $g(x)$ is a polynomial with integer coefficients. Hence, $f(10) = -21\, g(10)$, so that 21 must be a divisor of f (10). The only choice divisible by 21 is 42.

6. Find all values of a such that the quadratic equation $x^2 + (a - 3)x + a = 0$ has two distinct positive real solutions.

Answer: B
Solution: Since $-(a - 3)$ is the sum of the roots and a is the product of the roots and since both are positive, we get $0 < a < 3$. Since the roots are real and distinct we know that the discriminant is > 0 . $(a - 3)^2 - 4a > 0$; $a^2 - 10a + 9 > 0$; $(a - 1)(a - 9) > 0$; $a < 1$ or $a > 9$ The condition $0 < a < 3$ now implies $0 < a < 1$

7. If f (x) satisfies $2f(x) + f(1 - x) = x^2$ for all x, then f (x) =

Answer: D

Solution: The given equation $2f(x) + f(1 - x) = x^2$ holds for all x. In particularly, the equation holds if we replace x with 1 - x. Thus, we deduce that $2f(1 - x) + f(x) = (1 - x)^2$. Subtracting this equation from twice the given equation results in

$$3f(x) = 2x^2 - (1 - x)^2 \qquad f(x) = \frac{x^2 + 2x - 1}{3}$$

8. What is the domain of f if $f(x) = \dfrac{1}{|x| - 1}$?

Answer: E

Solution: When the domain of a function is not specified it is assumed to be all real numbers except those for which the function is meaningless. If $|x| - 1 = 0$ (x = 1, x = - 1), then the function is undefined.

9. If $f(x) = |x - 1|$ and $g(x) = 1 - x^2$, then which of the following is $3f(-2) + 4g(-3)$?

Answer: A

Solution: $3f(-2) + 4g(-3) = 3|-2 - 1| + 4(1 - (-3)^2) = 9 - 32 = -23$

10. Suppose that $f(n) = \log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdots \log_{(n-1)} n$. Then the value of $\sum_{k=2}^{10} f(2^k)$ is what?

Answer: D

Solution: $f(n) = \log_2 3 \cdot \dfrac{\log_2 4}{\log_2 3} \cdot \dfrac{\log_2 5}{\log_2 4} \cdot \ldots \cdot \dfrac{\log_2 n}{\log_2 (n-1)} = \log_2 n$

$$\sum_{k=2}^{10} f(2^k) = \sum_{k=2}^{10} \log_2(2^k) = \sum_{k=2}^{10} k = 2 + 3 + 4 + \cdots + 10 = 54$$

11. Let s (*n*) denote the sum of the digits on *n*. For example, s(197) = 1 + 9 + 7 = 17. Let $s^2(n) = s(s(n))$, $s^3(n) = s(s(s(n)))$, and so on. What is the value of $s^{1999}(1999)$?

Answer: C

Solution: A pattern develops. $s(1999) = 28$, $s^2(1999) = s(28) = 10$, $s^3(1999) = s^2(28) = s(10) = 1$ $\quad s^4(1999) = s^3(28) = s^2(10) = s(1) = 1$ In other words, $s^k(1999) = 1$ for all $k \geq 3$

12. Given that $f(x) = (x^5 - 1)(x^3 + 1)$, $g(x) = (x^2 - 1)(x^2 - x + 1)$, and h(x) is a polynomial such that f(x) = g(x) h(x), what is the value of h(1)?

Answer: D

Solution: Since

$$h(x) = \frac{f(x)}{g(x)} = \frac{(x^5 - 1)(x^3 + 1)}{(x^2 - 1)(x^2 - x + 1)} = \frac{(x - 1)(x^4 + x^3 + x^2 + x + 1)(x^3 + 1)}{(x - 1)(x + 1)(x^2 - x + 1)} = x^4 + x^3 + x^2 + x + 1$$

h (1) = 5

13. Which of the following statements does NOT describe a function?

Answer: E
Solution: Choices A - D establish ordered pairs by assigning the elements of one set to the elements of a second set. In each of these relations, each element of the domain has only one element of the range assigned to it.

14. Which of the following sets of ordered pairs is a function?

Answer: A
Solution: Choices B - D yield more than 1 y value for each x value.

15. Which of the following statements describe the domain of the function defined by:

$$f(x) = \frac{\sqrt{x-3}}{(x-2)(x-\sqrt{3})}$$

Answer: B
Solution: 2 and $\sqrt{3}$ must be excluded from the domain because they lead to division by zero. Similarly, all real numbers less than 3 must be excluded because they yield negative values for x - 3 and thus involve square roots of negative numbers.

16. Which of the following terms describe(s) f (x) = 0 ?

Answer: C
Solution: The zero function assigns 0 to every member of its domain given by f(x) = 0. Any function which assigns the same constant to every member of its domain is a constant function. The identity function is f(x) = x

17. If f (x) = 3x + 2, what is the inverse of f?

Answer: C
Solution: An inverse of a function is found by interchanging the x and y and solving the equation back for y. $f(x) = 3x + 2 \quad x = 3y + 2 \quad g(x) = \frac{1}{3}x - \frac{2}{3}$

18. Select the TRUE statement about relations and functions from the following:

Answer: C

19. The domain of a function is

Answer: D

20. Which of the following is a feature of the relation $|x| + |y| = 6$?

Answer: A

21. Given F(x) = 2x - 1 and G(x) = $\frac{5}{x}$. If the inverse of G [F(x)] is K(x) the value of K(4) is

Answer: B

Solution: G [F(x)] = $\frac{5}{2x-1}$ The inverse is $x = \frac{5}{2y-1}$. Evaluating when x = 4, yo get y = $\frac{9}{8}$

22. The composition of a function and its inverse

Answer: B

23. Find an expression for f(4x) in terms of f(x), given that f(x) = $\frac{x}{x-1}$.

Answer: B

Solution: $$f(4x) = \frac{4x}{4x-1} = \frac{\frac{4x}{x-1}}{\frac{4x-1}{x-1}} = \frac{4\frac{x}{x-1}}{4\frac{x}{x-1} - \frac{1}{x-1}} = \frac{4f(x)}{3\frac{x}{x-1} + \frac{x-1}{x-1}} = \frac{4f(x)}{3f(x)+1}$$

24. The roots of f(x) = $x^3 - 12x^2 + 37x - b$ form an arithmetic progression when b is

Answer: A
Solution: Call the roots a - d, a, a + d. From the relationship between roots and coefficients, we know that b is the product of the roots: b = a(a+d)(a-d) = a($a^2 - d^2$). We also have
12 = a +(a + d) + (a - d) = 3a and 37 = a(a-d) + a(a + d) + (a+d)(a-d) = $3a^2 - d^2$ So, a = 4, a^2 = 16, and d^2 = 11, so that b = 20

25. Given the function f(x) satisfying f(x) + $2f(\frac{1}{1-x})$= x, find f (2).

Answer: B
Solution: We use the reversal trick twice:

$$f(x) + 2f(\frac{1}{1-x}) = x \qquad f(\frac{1}{1-x}) + 2f(1-\frac{1}{x}) = \frac{1}{1-x} \rightarrow 2f(\frac{1}{1-x}) + 4f(1-\frac{1}{x}) = 2(\frac{1}{1-x})$$

$f(1-\frac{1}{x}) + 2f(x) = 1 - \frac{1}{x} \rightarrow 4f(1-\frac{1}{x}) + 8f(x) = 4 - \frac{4}{x}$ Subtracting and adding to cancel the extraneous, you get 9 f(x) = $4 - \frac{4}{x} - 2(\frac{1}{1-x}) + x$ So 9f(2) = 4 - 2 + 2 + 2 = 6 → f(2) = $\frac{2}{3}$

26. Which of the following statements is false?

Answer: D
Solution: The product of the zeros of "f" = - d. So the product of the zeros is 0.

27. In interval notation, what is the range of the function f(x) = $\begin{cases} 2x+3 & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x < 2 \\ 3 & \text{if } x \geq 2 \end{cases}$

Answer: A
Solution: graph the picture and it can be seen that the range is y < 4

28. The graph of f(x) = $\frac{x^2 - x - 2}{x + 2}$ has an oblique asymptote. The equation of this asymptote is:

Answer: D
Solution: divide the denominator into the numerator and disregard the remainder and you'll get y = x - 3

29. Given f(x) = $\frac{1}{x-2}$ and g(x) = $\sqrt{x+3}$, where f(x) and g(x) are real-valued functions. Find the domain of g(f(x)).

Answer: E
Solution: The domain of f(x) is the x $\neq$ 2. The domain of g(x) is that x $\geq$ -3. The domain of the composition is when $\frac{1}{x-2} \geq -3$ Solving $\frac{3x-5}{x-2} \geq 0$ you get x > 2 or x $\leq \frac{5}{3}$

30. If f(x) = 1 + $\frac{1}{x}$, find f (f (f (x)))

Answer: C
Solution: f (f (f (x))) = f (f (1 + $\frac{1}{x}$)) = f ($\frac{2x+1}{x+1}$) = $\frac{3x+2}{2x+1}$

Tiebreakers

T1. For what real number(s) m is the graph of $x^2 + y^2 - 4x - 6y = m - 13$ tangent to the graph of $x + y = 5 + \sqrt{2m}$

Answer: m > 0
Solution: The quadratic is an equation of a circle $(x-2)^2 + (y-3)^2 = m$. Since the slope of the tangent line is m = -1, the equation of the radius through the point of tangency is x - y = -1.

Solving x - y = - 1 and $x + y = 5 + \sqrt{2m}$ give $x = 2 + \frac{\sqrt{2m}}{2}$ and $y = 3 + \frac{\sqrt{2m}}{2}$. Substituting into the circle equation you get m = m which is true for all m > 0

T2. Find the inverse of the function: $f(x) = \log_2 \frac{2x-1}{2}$.

Answer: $g(x) = \frac{2^{x+1}+1}{2}$ or y = 2^x = ½

Solution: $f(x) = \log_2 \frac{2x-1}{2}$ becomes $x = \log_2 \frac{2y-1}{2}$; $2^x = \frac{2y-1}{2}$; $y = \frac{2^{x+1}+1}{2}$

T3. What is the period of the graph of $y = 4\cos^3 x \sin x - 4\cos x \sin^3 x$?

Answer: $\frac{\pi}{2}$

Solution: $4\cos^3 x \sin x - 4\cos x \sin^3 x = 2\cos^2 x\,(2\cos x \sin x) - 2\sin^2 x\,(2\cos x \sin x) =$
$2\cos^2 x \sin 2x - 2\sin^2 x \sin 2x = 2 \sin 2x\,(\cos^2 x - \sin^2 x) = 2 \sin 2x \cos 2x = \sin 4x$ Period = $\frac{2\pi}{4} = \frac{\pi}{2}$