1993 MU ALPHA THETA NATIONAL CONVENTION

Analytic Geometry Answers.

1. C
2. A
3. D
4. B
5. D
6. A
7. C
8. C
9. D
10. C
11. D
12. A
13. C
14. B
15. C
16. D
17. C
18. A
19. C
20. C
21. B
22. E
23. D
24. D
25. E
26. A
27. C
28. B
29. C
30. D

ANALYTIC GEOMETRY SOLUTIONS (1993 NATIONAL CONVENTION)

1. Plot the points; the fourth point must be $(3,5)$ to make the edges have slopes of 1 and $1/3$.

2. The sides of the square have length $\sqrt{2}$ by the distance formula. Thus the radius is $\sqrt{2}/2$.

3. The perpendicular line has slope $-\frac{1}{m} = -\frac{4}{3} = y+2$, hence $y = -\frac{10}{3}$.

4. We want the perpendicular bisector of the line joining the two points; it has slope $-\frac{1}{m} = 2$ and goes through the midpoint $(3,3)$, hence is $y - 3 = 2(x-3)$.

5. The distance of a point (u,v) from a line $ax + by + c = 0$ is $\frac{|au+bv+c|}{\sqrt{a^2+b^2}}$. Thus we solve $\frac{12v-20}{13} = \pm 4$ to get the possible y-coordinates $v = 6,\ -\frac{8}{3}$.

6. The radius r is the area divided by the semiperimeter $s = (a+b+c)/2$. Thus $r = \frac{24}{(6+8+10)/2} = 2$.

7. The radius equals $-1-(-3) = 2$, hence the circle is $(x-2)^2 + (y+3)^2 = 4$.

8. By definition, the eccentricity is $1/2 < 1$, hence it is an ellipse.

9. Draw a picture. The wheel is a circle with radius r and center (r,r), passing through the point $(2,4)$. Hence $(2-r)^2 + (4-r)^2 = r^2$, giving $r = 2, 10$. Only 10 makes sense in the physical problem.

10. Equating the circles, $x^2 + y^2 + 2x + 4y - 8 = x^2 + y^2 - 6x - 6y + 10$ yields $8x + 10y = 18$.

11. Use the formula, $\tan\alpha = \dfrac{|m_2 - m_1|}{1 + m_1 m_2}$, with $m_1 = -\frac{2}{3}$ and $m_2 = -\frac{8}{3}$.

12. The general form is $y^2 = 4px$ or $y^2 = -4px$, where p is the distance from the vertex to the focus, or equivalently, the distance from the vertex to the directrix. In this case, the focus is at $(-3,0)$ and we choose the minus sign.

13. The tetrahedron has volume 1/6 the volume of the parallelopiped generated by the points, hence volume
$$\frac{1}{6}\begin{vmatrix} -1 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 3 & -2 \end{vmatrix} = \frac{14}{6} = \frac{7}{3}$$

14. The sum of the distances is $16 = 12 + 4$, so the vertices are at $(\pm 8, 0)$. For the minor axis, we use the Pythagorean theorem: $b^2 + 6^2 = \left(\frac{16}{2}\right)^2$, so $b^2 = 28$. Thus the ellipse is $\frac{x^2}{64} + \frac{y^2}{28} = 1$.

15. The discriminant $B^2 - 4AC = -12 < 0$, hence it is an ellipse.

16. None of the coordinates work in the equation. But $[\sqrt{2}, \frac{2\pi}{3}]$ can also be written $[-\sqrt{2}, -\frac{\pi}{3}]$, which does satisfy the equation.

17. In Cartesian coordinates, this is the triangle formed by $x = 2$, $y = x$ and $y = 0$, which has area 2.

18. Expanding the sine and converting to Cartesian coordinates yields $x^2 + y^2 = \sqrt{2}x + \sqrt{2}y$, a circle centered at $(\sqrt{2}/2, \sqrt{2}/2)$.

19. The sum of the squares of the direction cosines is one, hence $\cos^2 \frac{\pi}{3} + \cos^2 \frac{2\pi}{3} + \cos^2 \theta = 1$, which yields $\cos \theta = \sqrt{2}/2$ and $\theta = \pi/4$.

20. The distance between the centers minus the radii is $2\sqrt{2} - 1 - 1$.

21. Proceeding as in Problem 10, the chord is found to be $x + y = 8$. Setting $y = 8 - x$ in the equation for either of the circles, gives the endpoints $(1, 7)$ and $(6, 2)$. The distance between them is $5\sqrt{2}$.

22. None of the points works since the Pythagorean theorem never holds.

23. The solution, $y^2 = 12x$, is obtained as in Problem 12.

24. The easiest solution is to just check the three points in each equation. The only one in which they all work is $x + z = 2$.

25. The region is an equilateral triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Since the side lengths are $\sqrt{2}$, the area is $\frac{\sqrt{3}}{4}(\sqrt{2})^2 = \frac{\sqrt{3}}{2}$.

26. The plane of the base is $x + 2y + 2z - 2 = 0$. The distance from the vertex to the plane is obtained by the 3-dimensional analog of the formula in Problem 5: $\frac{|1 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 - 2|}{\sqrt{1^2 + 2^2 + (-2)^2}} = 1$.

27. The third equation equals twice the first equation minus the second, hence the intersection is the same as the intersection of the first two (nonparallel) planes, namely a line.

28. The general form is $\frac{y^2}{a^2} - \frac{x^2}{c^2 - a^2} = 1$. The given vertices yield $a = 2$. The eccentricity $e = 2 = \frac{c}{a}$, gives us $c = 4$, and therefore the equation becomes $\frac{y^2}{4} - \frac{x^2}{12} = 1$.

29. The point we want is the intersection of the perpendicular bisectors of the line segments between the points. For $(4,2)$ and $(-3,1)$, this is the line $y - 3/2 = -7(x - 1/2)$, or $y = -7x + 5$. For $(-3,1)$ and $(-2,-6)$, the line is $y + 5/2 = \frac{1}{7}(x + 5/2)$, or $7y = x - 15$. The intersection of these lines is $(1,-2)$.

30. The line has the form $y = -2x + b$, giving an area of $\frac{1}{2}b \cdot \frac{b}{2}$. Thus $b = \pm 2$, the negative being the only choice among the answers.