Mu Alpha Theta National Convention 2003
State Calculus Bowl Solutions

The correct answer to each question is given immediately after the question number in parentheses.

1. (33/2) To be continuous and differentiable, the $f(x)$ and $f'(x)$ must be the same for both sides of $x=2$ and $x=5$. This leads to $4+2A+B=1$, $4+A=1/2$, $2=25+5C+D$, and $1/4=10+C$. So $A=-3.5$, $B=4$, $C=-9.75$, and $D=25.75$. The sum is 16.5.

2. (-180) Light coming into a parabolic mirror parallel to the axis of symmetry will reflect toward the focus. The line $4x+3=0$ intersects the mirror at $(-3/4, 9/16)$. The focus of the mirror is at $(0, 1/4)$. So the line containing these two points is $5x+12y-3=0 \rightarrow ABC=-180$. Once this light gets to the other side of the mirror, it will reflect away from the mirror parallel to the axis of symmetry. Because of this, the E term will be 0, and $DEF=0$. So $ABC+DEF=-180$.

3. $(2e+2\pi)$ $A=\int_0^1 (1-x+x^2-x^3+\cdots)dx = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots = (\ln(x+1))]_0^1 = \ln 2 \rightarrow e^A = 2$

$B=\int_0^1\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)dx = x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots = (e^x-1]_0^1 = (e-1) \rightarrow 2B = 2e-2$

$C=\int_0^1 (1-x^2+x^4-x^6+\cdots)dx = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots = (\tan^{-1}x]_0^1 = \frac{\pi}{4} \rightarrow 8C = 2\pi$. So $e^A+2B+8C = 2e+2\pi$.

4. (2,2,-2,-2) $y=A\cosh x+B\sinh x+C\cos x+D\sin x=0=A+C$; $y'=A\sinh x+B\cosh x-C\sin x+D\cos x=0=B+D$; $y''=A\cosh x+B\sinh x-C\cos x-D\sin x=4=A-C$; and $y'''=A\sinh x+B\cosh x+C\sin x-D\cos x=4=B-D. \rightarrow (2,2,-2,-2)$

5. $(e^{\pi/2})$ Let $u=\ln x \rightarrow e^u = x \rightarrow e^u du = dx$. This changes the integrals to $\int e^u \sin u + e^u \cos u du$. This can be solved by parts and then replacing x gives: $(x\sin(\ln x)]_1^{e^{\pi/2}} = e^{\pi/2}$.

6. $(8\pi^3+118\pi^2+168\pi)$ The regions intersect at $x=-\pi$, 0, and π. The region is an odd function, so we can calculate only the part fo $x>0$ and then double it. $A=2\int_0^{\pi}\pi\left[\left(\frac{x}{\pi}+\sin x\right)^2-\left(\frac{x^3}{\pi^3}\right)^2\right]dx = 2\left(\frac{x^3}{3\pi}-2x\cos x+2\sin x-\frac{x^7}{7\pi^5}\right]_0^{\pi}+\pi^2 = \frac{29\pi^2+84\pi}{21} \rightarrow 42A = 58\pi^2+168\pi$.

$B=2\int_0^{\pi}2\pi x\left(\frac{x}{\pi}+\sin x-\frac{x^3}{\pi^3}\right)dx = \left(\frac{4}{3}x^3-4\pi x\cos x+4\pi\sin x-\frac{4x^5}{5\pi^2}\right]_0^{\pi} = \frac{8\pi^3+60\pi^2}{15} \rightarrow 15B = 8\pi^3+60\pi^2$. So $42A+15B = 8\pi^3+118\pi^2+168\pi$.

7. (4π) The trick to this question is to notice that the first and third terms in the integrand are odd functions. Thus they contribute nothing to the definite integral. Thus the question reduces to $-\int_{-\pi}^{\pi} x^2\cos x dx$. This integrates to $(-x^2\sin x-2x\cos x+2\sin x]_{-\pi}^{\pi} = 4\pi$.

8. (381/41) Curvature is defined as $\kappa = |y''|/(1+(y')^2)^{3/2}$. Radius of curvature is just the inverse. For the first problem, I found it simpler to convert the equation to the form: $16(x+3)^2+9(y+4)^2=144 \rightarrow 32(x+3)dx+18(y+4)dy=0$. Then $dy/dx=-16(x+3)/[9(y+4)] = -4(x+3)/(3\sqrt{-x^2-6x})$ when you substitute for y. The second derivative is then $d^2y/dx^2 = 12/(-x^2-6x)^{3/2}$. Plugging in for the radius of curvature, $1/\kappa = (7x^2+42x+144)^{3/2}/324$. For $x=-3$, this gives $1/\kappa = 9/4 \rightarrow 4|A| = 9$. For the second problem you get, $16(x+3)^2-9(y+4)^2=144 \rightarrow 32(x+3)dx-18(y+4)dy=0$. Then $dy/dx = 16(x+3)/[9(y+4)] = 4(x+3)/(3\sqrt{x^2+6x})$ when you substitute for y. The second derivative is then $d^2y/dx^2 = -12/(x^2+6x)^{3/2}$. Plugging in for the curvature, $\kappa = 324/(25x^2+150x+144)^{3/2}$. For $x=-3+3\sqrt{2}$, this gives $\sqrt{41}|\kappa| = 12/41 = \sqrt{41}|B|$. So $4|A|+\sqrt{41}|B| = 9+12/41 = 381/41$.

9. (2387) $x=(t-1)(t-4)^4 \rightarrow v=4(t-1)(t-4)^3+(t-4)^4 \rightarrow v=(t-4)^3(5t-8) \rightarrow a=3(5t-8)(t-4)^2+5(t-4)^3$ $\rightarrow a=(t-4)^2(15t-24+5t-20)=(t-4)^2(20t-44)$. The first time the particle will be at rest is at $t=8/5=A$. The particle moves to the left for t between 8/5 and 4. It reaches zero acceleration during this period at t of 11/5. It's speed at this point is $2187/125=B$. So $125(A+B)=125(8/5+2187/125)=2387$.

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10. $(14399/1200)$ $y=\ln x \to dy = dx/x$. For $x=1$ and $dx=1/10$, $y=0$ and $dy=1/10$. $y=\sqrt{x} \to dy = dx/\left(2\sqrt{x}\right)$. For $x=25$ and $dx=-1$, $y=5$ and $dy=-1/10$. $y=x^{1/3} \to dy = dx/\left(3x^{2/3}\right)$. For $x=125$ and $dx=-1$, $y=5$ and $dy=-1/75$. $y=x^{1/5} \to dy=dx/\left(5x^{4/5}\right)$. For $x=32$ and $dx=1$, $y=2$ and $dy=1/80$. Adding the y's and the dy's gives $14399/1200$.

11. $\left[32+9\sqrt{2}+9\ln\left(1+\sqrt{2}\right) \text{ OR } 32+9\sqrt{2}+9\sinh^{-1}1\right]$ $A=\int_0^3 dx\int_0^{1/2} dy \int_0^{1/4-y^2} dz = 3\left(y/4-y^3/3\right]_0^{1/2} = 1/4 \to 12A=3$ For surface area, let's get all the easy sides first. The side on $y=0$ has area 3/4. The side on $z=0$ has area 3/2. The areas on $x=0$ and $x=3$ are the same: $\int_0^{1/2} 1/4-y^2dy = 1/12$...times two is 1/6. Now for the surface area on the curved side. You need to calculate the length of the parabolic arc, L, and then multiply by 3 (the width) to find the area. $L=\int_0^{1/2}\sqrt{1+z'^2}\,dz=\int_0^{1/2}\sqrt{1+4y^2}\,dy$. Let $y=\frac{1}{2}\tan\theta \to dy = \frac{1}{2}\sec^2\theta d\theta$. This gets $L=\int_0^{\pi/4}\left(\sec^3\theta\right)/2\,d\theta$, which can be integrated by parts for $u=.5\sec\theta$ and $dv=\sec^2\theta d\theta$ to get $L=.25\left(\sec\theta\tan\theta+\ln\left|\sec\theta+\tan\theta\right|\right]_0^{\pi/4}$ $=0.25\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]$. Thus the area is three times this, and $12B$ is $29+9\sqrt{2}+9\ln\left(1+\sqrt{2}\right)$. So the answer is: $12(A+B)=32+9\sqrt{2}+9\ln\left(1+\sqrt{2}\right)$ OR $12(A+B)=32+9\sqrt{2}+9\sinh^{-1}1$ (same answer, just derived differently).

12. (B,E,A,C,D) Compare two functions by dividing them and looking at the limit as x goes to infinity. First off, C and D are much slower than A, B, or E. Now let's compare them. $\lim_{x\to\infty}(3/e)^x=\infty$, so C is faster than D. Now let's compare A and E.
$\lim_{x\to\infty}\frac{(x+1)^{x-1}}{x^x}=\lim_{x\to\infty}\frac{\left[(x+1)/x\right]^x}{x+1}=\frac{e}{\infty}=0$, so E is faster than A. Now let's compare B and E: $\lim_{x\to\infty}\frac{(x-1)^{x+1}}{x^x}=\lim_{x\to\infty}(x-1)\left(\frac{x-1}{x}\right)^x=\infty e=\infty$, so B is faster than E. Thus the answer is B, E, A, C, and D.

13. [(4,2)] $Area=r^2\theta/2$ and the $Perimeter=r\theta+2r=2Area/r+2r=32/r+2r\to dP=-32/r^2+2=0\to r=4$. Plugging back into the area equation gives $\theta=2$. So the required answer is (4,2).

14. (6π+24) The area of an astroid as given in the problem is $3\pi A^2/8$. The perimeter is $6A$. Plugging in the value of A given, you have that the $\text{Area} + \text{Perimeter} = 6\pi+24$. The derivation involves lots of using the sum of cosine squared and sine squared, as well as integration by parts, and the symmetry of the system (all of the calculations can be done in the first quadrant alone and then multiplied by 4).

15. (-15120) The constant term in the derivative will be the term that had a degree of 1 in $f(x)$. This is the fourth term. The rth term in the expansion of $(a+b)^n$ is $\binom{n}{r-1}(a)^{n-r+1}(b)^{r-1}=\binom{7}{3}(2x)^4\left(-\frac{3}{x}\right)^3=-15120x$. So the value of the constant term in the full expansion of $f'(x)$ is -15120.