1999 MU ALPHA THETA NATIONALS
TENNESSEE STATE BOWL
MU DIVISION

QUESTION # 1

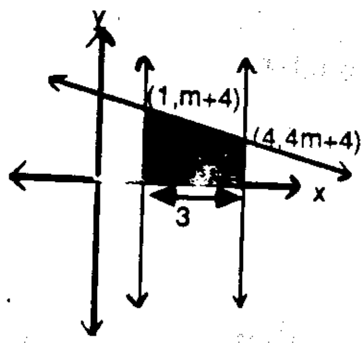
The convex region bounded by the x-axis and the lines with equations $y = mx + 4$, $x = 1$, and $x = 4$ has an area of 7. Find the value of m.

-2/3

The area in question is the shaded trapezoidal area indicated in the figure, with $h = 3$ and the lengths of the bases $m + 4$ and $4m + 4$.

In a trapezoid, $A = (1/2)h(b_1 + b_2)$

$7 = (3/2)(5m + 8)$
$14/3 = 5m + 8$
$5m = -10/3$
$m = -2/3$



QUESTION # 2

What is the work (in ft-lbs.) done by a force (in lbs.), propelling a particle along the x-axis from $x = 1$ to $x = 4$ (in feet), if $F = \frac{1}{\sqrt{x}}$?

2 ft-lbs

$$W = \int_1^4 x^{-1/2}dx = 2.$$

QUESTION # 3

Evaluate: $\int_0^{\frac{\pi}{4}} \tan^4 x \sec^4 x \, dx$

$\frac{12}{35}$

$$\int_0^{\frac{\pi}{4}} \tan^4 x \sec^4 x \, dx = \int_0^{\frac{\pi}{4}} \tan^4 x \left(1 + \tan^2 x\right) \sec^2 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^4 x \sec^2 x \, dx + \int_0^{\frac{\pi}{4}} \tan^6 x \sec^2 x \, dx$$

$$= \left(\frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x\right)\Big]_0^{\frac{\pi}{4}} = \frac{1}{5} + \frac{1}{7} = \frac{12}{35}$$

QUESTION # 4

A kite flies according to the parametric equations $y=\frac{-3}{64}t(t-128)$ and $x=\frac{t}{8}$, where t is measured in seconds and $0<t\le 90$. At what rate is the kite rising at t = 32 seconds?

3 $$\frac{dy}{dt}=\frac{-3}{32}t+6\Rightarrow @t=32,\frac{dy}{dt}=3.$$

5) If $\lim_{n\to\infty}\sum_{k=1}^{n}x^k=2$, then x equals?

$\frac{2}{3}$ The given problem is a geometric series and the Sum $=\frac{a}{1-r}$. S = 2, a = x, and r = x $\Rightarrow \frac{x}{1-x}=2\Rightarrow x=\frac{2}{3}$.

6) Evaluate: $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3\cos x+2x+3\right)dx$$

3π Note: If $f(x)$ is odd, $\int_{-a}^{a}f(x)\,dx=0$

So, $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3\cos x+2x+3\right)dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x^3\cos x\,dx+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2x\,dx+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}3dx=0+0+3x\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=3\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]=3\pi$$

A tank of 100 gallons of brine whose concentration is 2.5 lbs. of salt per gallon. Brine containing 2 lbs. of salt per gallon runs into the tank at a rate of 5 gallons/minute and the mixture (kept uniform) runs out at the same rate. What is the amount of salt in the tank as a function of time?

$y = 200 + 50e^{\frac{-t}{20}}$

$\frac{dy}{dt}$ = rate of inflow − rate of outflow = $10 - \frac{5y}{100}$. Using the initial conditions, we get $\frac{dy}{200-y} = \frac{dt}{20} \Rightarrow -\ln(200-y) = \frac{t}{20} + c \Rightarrow 200 - y = ke^{\frac{-t}{20}}$. Find k by substituting t = 0.

QUESTION # 8

For a certain curve, $\frac{dy}{dx} = \sqrt{xy - 12 - 3x + 4y}$.

The curve passes through the points (−3, 3) and (12, a). Find a.

444

$$\frac{dy}{dx} = \sqrt{(x+4)(y-3)}$$

$$(y-3)^{-\frac{1}{2}}\,dy = (x+4)^{\frac{1}{2}}\,dx$$

$$2(y-3)^{\frac{1}{2}} = \frac{2}{3}(x+4)^{\frac{3}{2}} + C$$

Passing through (−3, 3) implies $C = -\frac{2}{3}$.

So, when $x = 12$, $2(y-3)^{\frac{1}{2}} = \frac{2}{3}(16)^{\frac{3}{2}} - \frac{2}{3}$.

$$(y-3)^{\frac{1}{2}} = 21$$

$$y = 444$$

QUESTION 9A

The numbers 1, 3 = 1 + 2, 6 = 1 + 2 + 3, . . . are examples of triangular numbers. More generally, a positive integer is a triangular number if it can be written as the sum of the first n consecutive positive integers for some value of n. How many of the positive integers 1, 2, 3, . . ., 1999 are triangular numbers?

Answer: 62

Solution: Since the sum of the first n consecutive integers is $\frac{n(n+1)}{2}$ each n substituted will yield a triangular number. ∴ $\frac{n(n+1)}{2} \le 1999$ solving for n yields that n ≈ 62.7 and n ≈ -63.7 and since n has to be a natural number, n = 62.

9B

Find the value(s) of x for which the function

$f(x) = \ln\left[\ln\left(2x^2 - 16x + 33\right)\right]$ is discontinuous.

4 | For $f(x)$ to be continuous, $\ln\left(2x^2 - 16x + 33\right)$ must be greater than zero, which implies $2x^2 - 16x + 33$ must be greater than one. The minimum value of $2x^2 - 16x + 33$ occurs at $(4, 1)$.

QUESTION 10A

Find, in simplest fractional form, the cosine of the acute angle formed by the intersection of the asymptotes of the hyperbola with the equation:

$$16x^2 - 25y^2 + 32x + 100y = 484$$

$\frac{9}{41}$ |

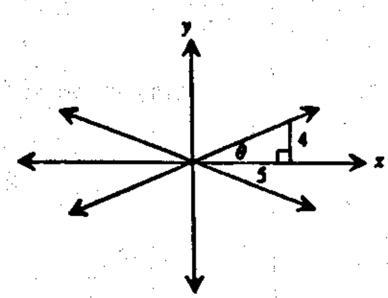
$$16\left(x^2 + 2x + 1\right) - 25\left(y^2 - 4y + 4\right) = 484 + 16 - 100$$

$$16(x+1)^2 - 25(y-2)^2 = 400 \Rightarrow \frac{(x+1)^2}{25} - \frac{(y-2)^2}{16} = 1$$

$$\Rightarrow \text{slopes of asymptotes} = \pm\frac{4}{5}$$

$$\cos\theta = \frac{5}{\sqrt{4^2+5^2}} = \frac{5}{\sqrt{41}}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{25}{41}\right) - 1 = \frac{9}{41}$$



QUESTION 10B

Find all real values of x which satisfy $x^{(x+1)^2} = x^{16}$

1, 0, −1, −5, 3 | Since the equation will be true when the base is 1, 0, or −1, these three values satisfy the equation. For any other values for the base, the exponents must be equal, so

$$(x+1)^2 = 16$$
$$x + 1 = 4 \text{ or } x + 1 = -4$$
$$x = 3 \text{ or } x = -5$$

Therefore, the 5 solutions are 1, 0, −1, 3, −5.

QUESTION 11D

A cylindrical can is undergoing a transformation in which the radius and height are varying continuously with time t. The radius is increasing at 4 inches/minute while the height is decreasing at 10 inches/minute. At what rate is the volume changing when the radius is 3 inches and the height is 5 inches?

30π

$$V = \pi r^2 h$$

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi r h \frac{dr}{dt}$$

$$= \pi(3)^2(-10) + 2\pi(3)(5)(4)$$

$$= -90\pi + 120\pi = 30\pi \text{ in}^3/\text{min}$$

QUESTION 12A

Let R be the region in the first quadrant that lies <u>below both</u> of the curves $y = 3x^2$ and $y = \frac{3}{x}$ and to the left of the line x = k where k > 1. Find the area of R as a function of k.

$(1 + 3\ln k)$

$$A = \int_0^1 3x^2 dx + \int_1^k \frac{3}{x} dx$$

QUESTION 12B

A spherical tank of radius 4 feet contains water to a depth of one foot. How much water must be added (exactly) to increase the depth by 1 foot?

$\frac{29\pi}{3}$

$$V = \pi \int_2^3 (\sqrt{16 - x^2})^2 dx$$

QUESTION 11A

How many distinct permutations of all the letters in the word PHOTOGRAPH are there such that the three vowels are adjacent and the seven consonants are also adjacent?

7560

$$\frac{7!}{2!\,2!} \cdot \frac{3!}{2!} \cdot 2! = 7560$$

QUESTION 11B

Find the maximum value of the <u>derivative</u> of

$$f(x) = -2x^3 + 12x^2 + 30x + 21$$

54

$$f'(x) = -6x^2 + 24x + 30$$
$$f''(x) = -12x + 24 = 0$$
$$x = 2$$

$$f'(2) = -6 \cdot 2^2 + 24(2) + 30$$
$$= -24 + 48 + 30$$
$$= 54$$

QUESTION 11C

The graphs of $y = 6 \sin x$ and $y = -3 \sin x$ intersect infinitely many times on the x-axis, bounding regions of equal areas. Find the area of one of these regions.

18

$$A = \int_0^{\pi} (6 \sin x - (-3 \sin x))\, dx$$
$$= \int_0^{\pi} 9 \sin x \, dx$$
$$= -9 \cos x \Big]_0^{\pi}$$
$$= -9(-1 - 1) = 18$$

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QUESTION 12C

Let f and g be functions defined on the positive integers and related in the following way:

$$f(n) = \begin{cases} 1, \text{ if } n = 1 \\ 2f(n-1), \text{ if } n \neq 1 \end{cases} \qquad g(n) = \begin{cases} 3g(n+1), \text{ if } n \neq 3 \\ f(n), \text{ if } n = 3 \end{cases}$$

What is the value of $g(1)$?

Answer: 36

Solution: $g(1) = 3g(2) = 3[3g(3)] = 9g(3) = 9[2f(2)] = 18f(2) = 18[2f(1)\} = 36f(1) = 36$

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QUESTION 12D

A particle moves along the curve $y = x^3 + 2$. Find the coordinates of all points on the curve at which the ordinate is changing 12 times as quickly as the abscissa.

(2, 10) (–2, –6)	Differentiating with respect to t: $\frac{dy}{dt} = 3x^2 \frac{dx}{dt}$. We want $\frac{dy}{dt} = 12 \frac{dx}{dt}$. so $3x^2 = 12$ $x^2 = 4$ $x = \pm 2$