Mu Alpha Theta National Convention: Seattle, 1997
Calculus State Bowl Test

1. Determine the sum: $\sqrt{4+\sqrt{7}}-\sqrt{4-\sqrt{7}}$.

2. A man goes into a flower shop to buy a certain number of flowers for an integer number of dollars. The clerk tells him of a special the store is having, where if he were to buy ten more flowers than he'd intended, he could have the entire purchase for two dollars, saving eighty cents a dozen. What was the price per flower, in cents, for the man's original purchase?

3. How many negative roots does the equation $x^4-5x^3-4x^2-7x+4=0$ have?

4. Determine the sum of $1(1!)+2(2!)+3(3!)+\cdots+n(n!)$

5. A triangular lake is enclosed by three square estates, having areas of 74, 116, and 370 acres, respectively. What is the acreage of the lake?

6. Determine all values of x for which $(6x+28)^{\frac{1}{3}} - (6x-28)^{\frac{1}{3}} = 2$

7. What is the smallest natural number composed of 4's and 6's but no other digits, such that both the number and the sum of its digits are both divisible by both 4 and 6?

8. Determine $v - w + x - y + z$ in the following system of equations.

$$v + w + y + z = 15$$
$$w + x + y + z = 24$$
$$v + w + x + y = 12$$
$$v + w + x + z = 8$$
$$v + x + y + z = 13$$

9. What is the least number greater than one which can be expressed as a square of a natural number, a triangular number, the sum of two triangular numbers, and the sum of three triangular numbers? The triangular numbers summed need not be distinct.

10. Trained ladybugs are placed at the vertices of a square 5 cm on a side. If each ladybug is trained to always walk straight towards the ladybug which starts on the vertex clockwise from itself, what will be the total distance walked by the four ladybugs after 30 minutes, if their walking speed is 1 cm/min ?

11. There are two bags, Bag 1 and Bag 2, each of which has 4 blue marbles, 4 red marbles, and 4 yellow marbles. I take a number of marbles from Bag 1 which is the minimum necessary to ensure I have at least two matching marbles and a marble of another color. These marbles I place in Bag 2. Then I transfer from Bag 2 to Bag 1 a number of marbles which is the minimum needed to ensure that Bag 1 will contain at least three marbles of each of the three colors. How many marbles will be in Bag 1 once this is done?

12. N is a seven digit number which ends in 7. M is its fifth root, which is a natural number. what is the value of M?

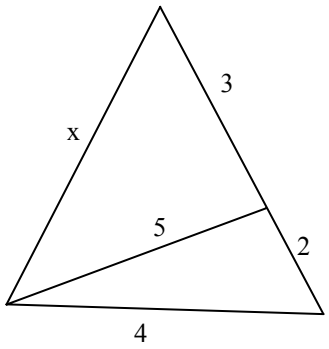
13. A sequence is defined as:

$$F_1 = 3$$
$$F_2 = 7$$
$$F_n = \frac{5F_{n-1} + 3F_{n-2}}{8} \quad for\ n > 2$$

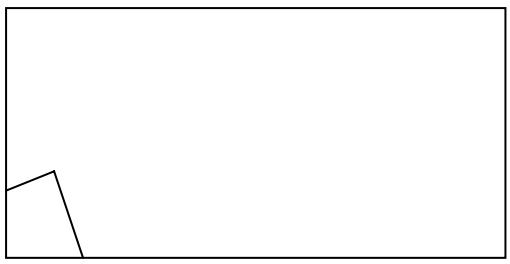
What is the limit of this sequence as n becomes large?

14. If $n(q) = \sqrt{3q + 14}$ and $p(n(q)) = 9q^2 + 87q + 210$, what is the value of $p(2)$?

15. What is the length of x in the triangle below?



16. What is the maximum area which can be enclosed in the corner of a rectangular room by two screens, one 3 meters wide, the other 6 meters wide?



17. Evaluate: $\int_{-2}^{0} \left(x^5 + 5x^4 + 10x^3 + 10x^2 + 7x + 8\right) dx$

18. Three distinct numbers are chosen from the first nine natural numbers. What is the probability that their sum is 15?

19. Supply the next number in the sequence:

18, 20, 22, 24, 30, 33, 102, 200, ___.

20. Evaluate: $\int_{0}^{1} x^4 e^x$

21. The sides of a cube are each painted one of 2 colors. How many distinguishable paintings are possible?

22. Let A be the smallest number half of which is a square, a third of which is a cube, and a fifth of which is a fifth power. How many factors does A have?

23. A woman typically takes the train home from work, arriving at the station at 5 o'clock, just as her husband arrives in the car to pick her up. One day the woman is able to leave work early and arrives at the station at 4 o'clock. Rather than calling her husband, she walks along the route she knows he will take, meeting him en route and arriving home twenty minutes earlier than she usually does. For how many minutes did the woman walk?

24. You have two bags of pebbles. Bag 1 contains 8 white and 5 black pebbles, while bag 2 contains 7 white and 2 black marbles. A marble is taken from bag 1 and placed in bag two. If a marble then drawn from bag 2 is black, what is the probability that the marble transferred from bag 1 was also black?

25. A sector with an acute central angle of 30º is cut from a circle of radius r. A circle of radius R is then circumscribed about that sector. What is the ratio of R to r?

26. If $x = \frac{1}{3}$, determine the value of $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x^{3n+m}$.

27. A wooden cube of edge length 16 has a square hole, whose sides are aligned with those of the cube drilled perpendicularly through the center of each pair of opposing faces. If the edge length of these holes is 9, what is the total surface area of the sculpture, including interior surfaces?

28. A each member of a yacht club pitches in to buy a dinghy. If the club had had 4 more members, the cost per person would have been 13 dollars less. If the club had had 4 fewer members, the cost per person would have been 17 dollars more. What was the price of the dinghy?

29. If a, b, c, and d are all non-zero real numbers such that a and b are the solutions to $x^2 + cx + d = 0$ and c and d are the solutions to $x^2 + ax + b = 0$, what is the sum of a, b, c, and d ?

30. What is the radius of the circumscribed circle of a triangle with sides of 8, 10 and 12?

31. Mr. MacGregor is expanding his garden. This year it contains 2245 more cabbages than it did last year, though he grows a square number of cabbages each year. What is the minimum number of cabbages he could be growing this year?

32. Evaluate: $\int_0^{\pi} \sin^4 3x$

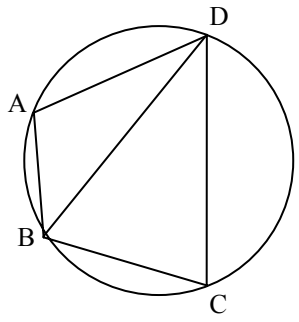
33. An ant is on the "equator" of a child's top composed of two cylinders of height $\sqrt{95}$ cm and base radius 7 cm joined at their bases. If the ant wishes to get to the point on the "equator" (the circle at which the two bases are joined) 180º away from his current position, what is the shortest distance he can travel?

34. What is the area in the first quadrant bounded by the line $y = x$, the polar curve $r = \cos\theta$, and the x-axis?

35. What is the volume of the solid generated when the region bounded by $y = -(x+16)^2 + 16$ and the x-axis is rotated about the y-axis?

36. How many times each day does the smaller angle between the second hand and the hour hand equal the smaller angle between the second hand and the minute hand?

37. Convex quadrilateral ABCD is inscribed in a circle. If AB is 4, BC is 19, CD is 16, and AD is 11, what is the length of BD?



38. Determine the maximum value of $f(x) = 2x^3 - 21x^2 + 7x + 12$ on the closed interval from 1 to 5.

39. What is the area of a regular dodecagon with perimeter 144?

40. Evaluate: $\dfrac{\left(4+\sqrt{15}\right)^{\frac{3}{2}} + \left(4-\sqrt{15}\right)^{\frac{3}{2}}}{\left(6+\sqrt{35}\right)^{\frac{3}{2}} - \left(6-\sqrt{35}\right)^{\frac{3}{2}}}$

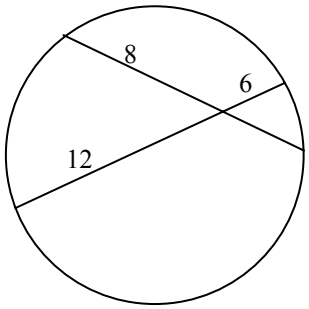
41. The smaller of two consecutive natural numbers is divisible by 49, and the larger by 53. Find the smallest possible value for the smaller of the two numbers.

42. A woman went out for lunch, noting the time as she left. When she returned, between one and two hours later, the hands of her clock had switched places. How many hours was she gone?

43. Tom is barbecuing some big burgers, which need precisely 8 minutes of barbecuing on each side. Each burger is so big and juicy, that it takes Tom an entire minute to carefully remove one from the grill to a nearby platter, or to move one from the platter to the grill. It also takes a full minute to flip one of the burgers, and only one action can be performed at a time. If his grill only holds two of his burgers at a time, what is the minimum number of minutes in which he can fully grill both sides of 3 burgers?

44. Evaluate: $\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-i}$

45. What is the radius of the circle below if the two chords shown are perpendicular?



46. Determine the sum of the eigenvalues of the matrix $\begin{bmatrix} 3 & 4 & 2 \\ 4 & -1 & 2 \\ 1 & 5 & 5 \end{bmatrix}$.

47. What is the equation of the plane through the points (2, 3, 4), (1, 4, -2), and (-3, -6, 1)?

48. Find the sum of the three smallest natural numbers with exactly 12 factors, none of which are divisible by 6.

49. There are two towers on opposite ends of a flat field. Tower A is 100 meters tall, and when a woman stands at the base of the tower B, she must look up at an angle of 75º to see the top of tower A. when the woman ascends to the top of tower B, however, she must look down at an angle of 15º to see the top of tower A. What is the height of tower B?

50. A plane flew with a tailwind all the way from Arlando to Zeattle in 5 hours. It then flew back, this time with a headwind of equal magnitude to the earlier tailwind, and made the trip in 90 minutes more than it would have taken in still air. What is the longest that the return trip could have taken, in minutes?

51. What is the sum of the cubes of the roots of the equation $x^3 + 4x^2 - 3x + 2 = 0$?

52. How many 4 digit natural numbers have their digits in descending order from left to right?

53. A farmer notes that 24 cows graze 10 lush acres bare in 2 weeks, and 12 cows graze the same area bare in 6 weeks. How long would it take 15 cows to graze 20 acres bare?

54. Jon will arrive home from work sometime between 5 and 6 PM, stay for twenty minutes, then leave to have dinner. Joy will call Jon sometime between 5 and 6 PM, let the phone ring for five minutes, then give up. What is the probability that Joy will be calling while Jon is home?

55. Determine the sum of all values of θ $(0 \le \theta < 2\pi)$ for which $\cot^2 3x = \csc 3x + 1$.

56. What is the volume of the solid formed by the rotation of the region enclosed by the graph of $y = 4 - x^2$ and the x-axis about the y-axis?

57. Train A is 50 miles north of Chicago, heading west at 10 kmph. Train B is 20 miles east of Chicago, heading north at 30 kmph. Let θ be the angle relative to his motion that the conductor of train B must look to see train A. How rapidly is θ changing, in radians per second?

58. What is the probability that when a knight and king of opposite colors are randomly placed on a chessboard, the king is in "check"?

59. How many positive integers are less than and relatively prime to 210?

60. In how many ways can one trace a path in 3-space from the origin to the point (4, 4, 6) where each leg of the path is between adjacent lattice points, and the direction of travel is always parallel to either the x, y, or z axes?