Mu Alpha Theta National Convention 2003
Calculus Sequences and Series Test

Solutions:

1. $(1)(1.1)^{n-1} \geq 5 \Rightarrow n-1 \geq \frac{\ln 5}{\ln 1.1}$. $n-1 \geq 16.886$ so n=18. **Choice D.**

2. $\sqrt{175 \bullet 567} = 315$. **Choice A.**

3. 11(3+5+7+9)=264. **Choice B.**

4. $\frac{20}{2}(a_1 + 79) = 820 \Rightarrow a_1 = 3$. $3 + (20-1)d = 79$ so d=4. **Choice C.**

5. $\frac{\cos\theta}{1-\sin\theta} = \frac{\sqrt{1-\sin^2\theta}}{1-\sin\theta} = \sqrt{3}$. Square both sides: $\frac{1-\sin^2\theta}{1-2\sin\theta+\sin^2\theta} = 3$. $2\sin^2\theta - 3\sin\theta + 1 = 0$. Factor so $(2\sin\theta - 1) = 0$ or $(\sin\theta - 1) = 0$. In the second case, the common ratio is 1 and the series diverges. In the first case, $\theta = \pi/6$ *or* $5\pi/6$ but only $\theta = \pi/6$ yields a positive result for the cosine function. **Choice A.**

6. $2 \bullet \frac{n(n+1)(2n+1)}{6} + 18^2 = \frac{17(18)(35)}{3} + 324 = 3{,}894$. **Choice C**.

7. $\frac{4+i}{1-r} = 3+5i \Rightarrow 1-r = \frac{(4+i)(3-5i)}{(3+5i)(3-5i)} = \frac{17-17i}{34}$.

   $1-r = 0.5 - 0.5i$ so $r = 0.5 + 0.5i$. **Choice D.**

8. Pattern = Wed, Sat, Tues, Fri, Mon, Thurs, Sun, Wed…. So pattern repeats every 3 weeks. A 365-day year has 52 weeks and one day. Since 51 divided by 3 is 17, pattern will cycle 17 times plus Weds, Sat and Tues in the 52nd week. There are 52 Tuesdays in 2003 so the probability is 18/52=9/26. **Choice C.**

9. $4\sum_{n=3}^{12} 2^n + 6\sum_{n=3}^{12} n = 4\left(\frac{8 - 8 \bullet 2^{10}}{1-2}\right) + 6\left(\frac{10}{2}\right)(3+12) =$ $3{,}736 + 450 = 33{,}186$. **Choice B.**

10. $\sqrt{13 + \sqrt{5+x}} = x$. $\sqrt{5+x} = x^2 - 13$.
    $5 + x = x^4 - 26x^2 + 169 \Rightarrow 0 = x^4 - 26x^2 - x + 164$. Graph or test and find x=4. **Choice B.**

11. $\sum_{i=1}^{z}(6i-5) = 6\sum_{i=1}^{z} i - 5z = \frac{6z(z+1)}{2} - 5z = 40$.
    $3z^2 - 2z - 40 = 0 \Rightarrow (3z+10)(z-4) = 0$. The value of z must be positive so z=4. **Choice A.**

12. $C(power, n-1) = C(1.5,4) = \frac{\frac{3}{2} \bullet \frac{1}{2} \bullet \frac{-1}{2} \bullet \frac{-3}{2}}{4!} = \frac{3}{128}$. **Choice B.**

13. $S = \frac{17}{3} + \frac{13}{9} + \frac{9}{27} + ...; 3S = 17 + \frac{13}{3} + \frac{9}{9} + \frac{5}{27} ...$. So $3S - S = 17 - \left(\frac{4}{3} + \frac{4}{9} + \frac{4}{27} + ...\right) = 17 - \frac{4/3}{1-1/3} = 17 - 2 = 15$. S=15/2=7.5. **Choice B.**

14. $a_2 = 10^{10-9} = 10; a_3 = 10^{10-9} = 10$. All sequence terms are 10. **Choice B.**

15. $\lim_{n\to\infty} 4 + \frac{5}{n} = 4 + \frac{5}{\infty} = 4 + 0 = 4$. **Choice A.**

16. $\lim_{n\to\infty} \frac{n^3 - 2}{\sqrt{9n^6 + n^2}} \div \frac{n^3}{\sqrt{n^6}} = \lim_{n\to\infty} \frac{1 - 2/n^3}{\sqrt{9 + 1/n^4}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$. **Choice C.**

17. $\lim_{n\to\infty}\left(1 + \frac{1}{2n}\right)^n = \lim_{n\to\infty}\left(\left(1 + \frac{1}{2n}\right)^{2n}\right)^{0.5} = e^{0.5}$. **Choice A.**

18. The terms of the sequence will always be less than or equal to the terms of $\frac{n^{\frac{3}{2}}}{2n-5}$. $\lim_{n\to\infty} \frac{n^{\frac{3}{2}}}{2n-5} \div \frac{n}{n} =$ $\lim_{n\to\infty} \frac{\sqrt{n}}{2 - 5/n}$ which diverges. **Choice D.**

19. $f'(x) = 3(\ln x)^2 \frac{1}{x}$. $f''(x) = 6\ln x\left(\frac{1}{x}\right)\left(\frac{1}{x}\right) + \frac{-3(\ln x)^2}{x^2}$.
    $f'''(x) = 6\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)\left(\frac{1}{x}\right) + \frac{-12\ln x}{x^3} + \frac{-6\ln x}{x^3} + \frac{6(\ln x)^2}{x^3}$.
    $f'''(e^3) = \frac{6}{(e^3)^3} - \frac{12\ln e^3}{(e^3)^3} - \frac{6\ln e^3}{(e^3)^3} + \frac{6(\ln e^3)^2}{(e^3)^3} =$

$$\frac{6-12(3)-6(3)+6(3)^2}{e^9}=\frac{6-36-18+54}{e^9}=\frac{6}{e^9}.$$

Choice C.

20. $\dfrac{A}{n-2}+\dfrac{B}{n+1}=\dfrac{1}{(n-2)(n+1)}\Rightarrow A(n+1)+B(n-2)=1.$

Thus A+B=0 and A-2B=1. Solve to get A=1/3, B=-1/3.

$$\frac{1}{3}\sum_{n=5}^{\infty}\left(\frac{1}{n-2}-\frac{1}{n+1}\right)=\frac{1}{3}\left[\left(\frac{1}{3}-\frac{1}{6}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{5}-\frac{1}{8}\right)+\left(\frac{1}{6}-\frac{1}{9}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\ldots\right]$$

All terms will cancel except 1/3, 1/4, and 1/5. So the sum of these three terms is 47/60 and 1/3 of the sum is 47/180. **Choice B.**

21. $|a_{n+1}|<0.00005$ so $\dfrac{2}{(n+1)^2+4}<0.00005$ which yields $(n+1)^2>39{,}996$ and n>=199. **Choice B.**

22. $Q(b)=\left(\dfrac{\sqrt{b}}{2}\right)^2\pi=\dfrac{b\pi}{4}.\ Q\left(\dfrac{2}{p}\right)=\dfrac{\pi}{2p}.\ \displaystyle\sum_{p=1}^{\infty}Q\left(\frac{2}{p}\right)=$

$\dfrac{\pi}{2}\displaystyle\sum_{p=1}^{\infty}\frac{1}{p}$ which is the harmonic series. Series diverges.

Choice D.

23. $\dfrac{a_{n+1}}{a_n}=\dfrac{(n+1)^{(n+1)}}{(n+1)!}\div\dfrac{n^n}{n!}=\dfrac{(n+1)^{(n+1)}n!}{(n+1)!n^n}=\dfrac{(n+1)^{(n+1)}}{(n+1)n^n}=$

$\dfrac{(n+1)^n}{n^n}$ which is greater than one for all n>1. Thus the sequence is everywhere in creasing. **Choice A.**

24. $\left|\lim_{n\to\infty}\dfrac{a_{n+1}}{a_n}\right|<1.\ \lim_{n\to\infty}\dfrac{3(n+1)^{\frac{4}{5}}x^{n+1}}{3^{n+1}}\div\dfrac{3n^{\frac{4}{5}}x^n}{3^n}=\left(\dfrac{n+1}{n}\right)^{\frac{4}{5}}\dfrac{x}{3}$

$\left(\dfrac{n+1}{n}\right)^{\frac{4}{5}}\dfrac{x}{3}=\dfrac{x}{3}$. Thus $\left|\dfrac{x}{3}\right|<1$ and $|x|<3$. Check the endpoints. At x=3, $\lim_{n\to\infty}\dfrac{3n^{\frac{4}{5}}3^n}{3^n}=3n^{\frac{4}{5}}$ which diverges so the series diverges. At x=-3, the series is $3(-1)^n n^{\frac{4}{5}}$. We know the absolute value of the terms tends to infinity so by the alternate series test the series diverges. Exclude both endpoints so relevant interval is (-3,3). **Choice D.**

25. $\lim_{x\to\infty}\displaystyle\int_0^x -3n^2e^{-n^3}\,dn=\lim_{x\to\infty}e^{-n^3}\Big]_0^x=\lim_{x\to\infty}e^{-x^3}-e^0.$

$\lim_{x\to\infty}e^{-x^3}-1=0-1=-1.$ Because the limit converges, the series converges. **Choice A.**

26. $\lim_{x\to\infty}\sqrt[x]{\dfrac{1}{[\cos(5/x)]^x}}=\lim_{x\to\infty}\dfrac{1}{\cos(5/x)}=\dfrac{1}{\cos 0}=1.$ The series converges if the limit is less than 1 so this series diverges. **Choice C.**

27. $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\dfrac{(n+1)^{e^2}e^{-(n+1)^2}}{n^{e^2}e^{-n^2}}=$

$\lim_{n\to\infty}\left(\dfrac{n+1}{n}\right)^{e^2}e^{-(n+1)^2+n^2}=\lim_{n\to\infty}\left(\dfrac{n+1}{n}\right)^{e^2}\dfrac{1}{e^{2n+1}}=1\bullet 0=0.$

The Ratio Test says the series converges because the limit is less than 1. **Choice B.**

28. The test says in order for the series to converge (1) the absolute value of the terms decrease and (2) the sequence's terms must tend to 0. We can graph the sequence's numerator and denominator and note the denominator increases faster so (1) is true. For (2) use l'Hopital's rule: $\lim_{x\to\infty}\dfrac{\ln(s^2)}{4^s}=\lim_{s\to\infty}\dfrac{2s}{s^2\ln 4\bullet 4^s}$ which equals 0. Thus (1) and (2) are satisfied and the series converges. Choice B is incorrect because the limit must not only converge but be 0. **Choice E.**

29. f(0)=0. f'(x)=-2cos(x)sin(x)=-sin(2x) and f'(0)=0. f''(x)= -2cos(2x) so f''(0)=-2. f'''(x)=4sin(2x) so f'''(0)=0. $f^{IV}(x)$=8cos(2x) so $f^{IV}(0)$=8. $f^{V}(x)$=-16sin(2x) so $f^{V}(0)$=0. $f^{VI}(x)$=-32cos(2x) so $f^{VI}(0)$=-32. The series is:

$0+0x+\dfrac{2x^2}{2!}+\dfrac{0x^3}{3!}+\dfrac{8x^4}{4!}+\dfrac{0x^5}{5!}+\dfrac{-32x^6}{6!}+\ldots$. By inspection, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n2^{2n-1}}{(2n)!}x^{2n}$. **Choice D.**

30. $\displaystyle\sum_{i=1}^{20}\left(x^3\Big]_{i+1}^{i+3}\right)=\sum_{i=1}^{20}(i+3)^3-\sum_{i=1}^{20}(i+1)^3=$

$(4^3+5^3+6^3+\ldots+23^3)-(2^3+3^3+4^3+\ldots+21^3)=$

$22^3+23^3-2^3-3^3=22{,}780.$ **Choice C.**