# Mu Alpha Theta National Convention 2003
# Calculus Bowl Solutions

The correct answer to each question is given immediately after the question number in parentheses.

1. (3) I is false because $f'(x)$ is rising at A. II is false because $f''(x)$ is zero at B. III is false because the slope of $f'(x)$ is negat at C, so $f(x)$ is concave down. IV is true. V is false because $f''(x)$ is positive at D, not zero. VI is true. VII is false because the slope of $f'(x)$ is negative at G, implying that $f(x)$ has a relative maximum there. VIII is true. So there are three true statements.

2. $\left(-735\sqrt{2}/4\right)$ $f'(x)=2(8x)(4x^2+1)(2x-7)^3\cos^3(x-\pi/4)+3(2)(4x^2+1)^2(2x-7)^2\cos^3(x-\pi/4)-3\sin(x-\pi/4)(4x^2+1)^2(2x-7)^3\cos^2(x-\pi/4$

$f'(0)=0+6(1)(49)\left(\sqrt{2}/2\right)^3-3\left(-\sqrt{2}/2\right)(1)(-343)\left(\sqrt{2}/2\right)^2=-735\sqrt{2}/4$

3. (255) $\int_1^A\sqrt{w-1}\,dw=144=\left(\frac{2}{3}(w-1)^{3/2}\right]_1^A\rightarrow A=37$; $\int_{B/2}^B x^2+3x+4\,dx=690=\left(\frac{x^3}{3}+\frac{3x^2}{2}+4x\right]_{B/2}^B=\frac{7B^3}{24}+\frac{9B^2}{8}+2B\rightarrow B=12$;

$\int_1^C\frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}=\frac{2}{5}=\left(\frac{-1}{1+\sqrt{y}}\right]_1^C\rightarrow C=81$; and $\int_1^D(\log_5 z)^2\,dz/z=9\ln 5=\left[1/(\ln 5)^2\right]\int_1^D(\ln z)^2\,dz/z=\left[1/(\ln 5)^2\right]\left((\ln z)^3/3\right]_1^D\rightarrow D=125$.

So the sum of the four numbers is 255.

4. (108) Positions: M, $x=0$ and $y=12-12t$; A, $x=8t$ and $y=8t$; Th, $x=8-10t$ and $y=0$

Distance: MA, $s^2=(8t)^2+(20t-12)^2\rightarrow 2s\,ds/dt=128t+800t-480=0\rightarrow t=15/29$ (hrs) $\approx 31$ min $=A$,

Mθ, $s^2=(8-10t)^2+(12-12t)^2\rightarrow 2s\,ds/dt=160-200t+288-288t=0\rightarrow t=56/61$ (hrs) $\approx 55$ min $=B$, and

Aθ, $s^2=(18t-8)^2+(8t)^2\rightarrow 2s\,ds/dt=648t-288+128t=0\rightarrow t=36/97$ (hrs) $\approx 22$ min $=C$. So the sum of $A$, $B$, and $C$ is 108.

5. (76) $\int_{-4}^4\left|x^3-7x-6\right|dx=\int_{-4}^{-2}-(x^3-7x-6)dx+\int_{-2}^{-1}(x^3-7x-6)dx+\int_{-1}^{3}-(x^3-7x-6)dx+\int_3^4(x^3-7x-6)dx$

$=\left(-x^4/4+7x^2/2+6x\right]_{-4}^{-2}{}_{-1}^{3}+\left(x^4/4-7x^2/2-6x\right]_{-2}^{-1}{}_{3}^{4}=76$

6. $\left(-\pi^2\sqrt{2}/8+\pi\sqrt{2}\right)$ $\int_0^{2\pi}x\cos(x/2)dx=\left(2x\sin(x/2)+4\cos(x/2)\right]_0^{2\pi}=-8=A$, $\lim_{y\to 3}\frac{y^2+2y-15}{y^2-8y+15}=\lim_{y\to 3}\frac{y+5}{y-5}=-4=B$, and

$f(z)=z^2\cos(-z/4)\rightarrow f'(z)=z^2\left[-\sin(-z/4)\right](-1/4)+2z\cos(-z/4)\rightarrow f'(\pi)=-\pi^2\sqrt{2}/8+\pi\sqrt{2}=C$

7. (518) $(d/dw)\left[\int_0^w f(t)dt=w\cos(\pi w)\right]\rightarrow f(w)=-\pi w\sin(\pi w)+\cos(\pi w)\rightarrow f(4)=1=A$. $(d/dx)\left[\int_0^{x^2}g(t)dt=x\cos(\pi x)\right]=2xf(x^2)=$

$-\pi x\sin(\pi x)+\cos(\pi x)\rightarrow f(2^2)=1/4=B$. $h(y)=(d/dy)\left(\int_{-2y}^{y^2}t^3\sqrt{4+t}\,dt\right)=(2y)y^6\sqrt{4+y^2}-(-2)(-8y^3)\sqrt{4-2y}$. $h(2)=512\sqrt{2}=C$.

$\lim_{z\to 0}\frac{\int_0^{z^2}t^2\,dt/(t^4+1)}{z^6}=\frac{0}{0}$. Use L'Hopital's rule, $\lim_{z\to 0}\frac{2z^5/(z^8+1)}{6z^5}=\lim_{z\to 0}\frac{1}{3(z^8+1)}=\frac{1}{3}=D$ So $A+16B+C\sqrt{2}/2+3D=1+4+512+1=518$.

8. $\left[\left(12\sqrt{3}-5\right)L/11\right]$ Let $x$ be a side of the square and $y$ be a side of the triangle. Then $4x+3y=L\rightarrow y=(L-4x)/3$.

$Area=x^2+y^2\sqrt{3}/4=x^2+\left(\sqrt{3}/4\right)\left[(L-4x)/3\right]^2\rightarrow Ared'=2x+2\left[(L-4x)/3\right](-4/3)\left(\sqrt{3}/4\right)=0\rightarrow x=\left(3\sqrt{3}-4\right)L/11$.

$Ared''(x)=2+8\sqrt{3}/9>0\rightarrow$ minimum area $\rightarrow 4x=\left(12\sqrt{3}-16\right)L/11=A$. The maximum area forms when all of the area is in the square

Then the area is $(L/4)^2=L^2/16$ (as compared to $L^2\sqrt{3}/36$ for the triangle). $\therefore B=L$, $A+B=\left(12\sqrt{3}-5\right)L/11$.

9. (6) $\lim_{w\to 0}\tan 2w\csc 4w=\lim_{w\to 0}\frac{\sin 2w}{\cos 2w\sin 4w}=\lim_{w\to 0}\frac{2\cos 2w}{4\cos 2w\cos 4w-2\sin 2w\sin 4w}=\frac{1}{2}=A$.

$\lim_{x\to 6}\frac{x^4-2x^3-23x^2-12x+36}{x^4-6x^3-5x^2-106x+120}=\frac{0}{-696}=0=B$. $C=\lim_{y\to\infty}(3y^3+5)^{1/\sqrt[3]{y}}\rightarrow\ln C=\lim_{y\to\infty}\frac{\ln(3y^3+5)}{y^{1/3}}=\lim_{y\to\infty}\frac{9y^2/(3y^3+5)}{y^{-2/3}/3}=\lim_{y\to\infty}\frac{27y^{8/3}}{3y^3+5}=0\rightarrow C=1$.

$\lim_{z\to-\infty}\sqrt{z^2-5z}-\sqrt{z^2+7z}=\lim_{z\to-\infty}\left(\sqrt{z^2-5z}-\sqrt{z^2+7z}\right)\left(\frac{\sqrt{z^2-5z}+\sqrt{z^2+7z}}{\sqrt{z^2-5z}+\sqrt{z^2+7z}}\right)=\lim_{z\to-\infty}\frac{-12z}{\sqrt{z^2-5z}+\sqrt{z^2+7z}}=6=D$. So, $AB+CD=6$.

10. (A,D) $T=(h/2)(y_0+2y_1+2y_2+2y_3+2y_4+2y_5+y_6)=(1/6)(1+32/9+50/9+8+49/9+64/9+9)=235/27$., TRUE. $\int_0^{\pi/4}\tan^4 x\sec^4 x\,dx=\int_0^{\pi/4}\tan^4 x(\tan^2 x+1)\sec^2 x\,dx=\left(\tan^7 x/7+\tan^5 x/5\right]_0^{\pi/4}=12/35$, which means B is FALSE. The volume of a torus is the cross-sectional area times the length. In this case, the circle's area is $4\pi$. The length is $10\pi$. So the volume is $40\pi^2$, which means the C statement is FALSE. $5y^3=x^2$ intersects the circle $x^2+y^2=6$ where $5y^3=6-y^2\rightarrow 5y^3+y^2-6=0$, which has a root at $y=1$. The variable $x$ has two roots at $\pm\sqrt{5}$. From our original formula, we know that $dx/dy=3\sqrt{5y}/2$, so the arc length is $s=2\int_0^1\sqrt{1+(45y/4)}\,dy=\left((16/135)[1+(45y/4)]^{3/2}\right]_0^1=134/27$, which is TRUE.

11. $(10+1/e)$ $g(f(x))=x\rightarrow g'(f(x))f'(x)=1\rightarrow g'(f(x))=1/f'(x)$. For (I.), $f(x)=3$ when $x=1/2$. $f'(x)=-x^{-2}$ $\rightarrow g'(3)=1/f'(1/2)=-1/4=A$. For (II.), $f(x)=2$ when $x=1$. $f'(x)=5x^4+3x^2\rightarrow g'(2)=1/f'(1)=1/8=B$. For (III.), $f(x)=2$. when $x=5$ (since $x>2$). $f'(x)=2x-4\rightarrow g'(2)=1/f'(5)=1/6=C$. For (IV.), $f(x)=1$ when $x=e$. $f'(x)=x^{-1}\rightarrow g'(1)=1/f'(e)=e=D$. So $1/A+1/B+1/C+1/D=-4+8+6+1/e=10+1/e$.

12. (7226/3) $dy/dx=\sqrt{(x+3)(y-7)}\rightarrow dy/\sqrt{y-7}=\sqrt{x+3}\,dx\rightarrow 2\sqrt{y-7}=(2/3)(x+3)^{3/2}+C$. Plugging in (-2,8) gets a $C$=4/3. Plugging in 1, 2, 3, and 4 for $x$ gets 163/9, 904/9, 4419/9, and 16192/9, which has a sum of 7226/3.

$\int_1^4(u-1)\sqrt{u}/2\,du=\left(u^{5/2}/5-2u^{3/2}/3\right]_1^4=58/15$

13. $[(e^2+\pi)/4]$ Let $u=1+x^2$, then $du=2x\,dx$, and we have $\int_1^4(u-1)\sqrt{u}/2\,du=\left(u^{5/2}/5-2u^{3/2}/3\right]_1^4=58/15=A$. Let $u=x^2$, then $du=2x\,dx$ and $\int_0^{\pi/4}\frac{1}{2}\tan^2 u\,du=\int_0^{\pi/4}\frac{1}{2}(\sec^2 u-1)\,du=\left((\tan u)/2-u/2\right]_0^{\pi/4}=(1/2)-(\pi/8)=B$. $\int_3^5\frac{3x^2-6x-28}{x^3-3x^2-28x+60}dx=\int_3^5\frac{1}{x+5}+\frac{1}{x-2}+\frac{1}{x-6}dx$ $=\left(\ln|x+5|+\ln|x-2|+\ln|x-6|\right]_3^5=\ln\frac{5}{4}=C$. Let $u=(\ln x)^2$ and $dv=x\,dx$. Then $du=2\ln x\,dx/x$ and $v=x^2/2$. This makes the integral equal to $x^2(\ln x)^2/2-\int x\ln x\,dx$. Another int. by parts with $u=\ln x$ and $dv=x\,dx$ gives $du=dx/x$ and $v=x^2/2$. This makes the integral $x^2(\ln x)^2/2-(x^2\ln x)/2+\int x\,dx/2$. Solving for the definite integral gives $\left(x^2(\ln x)^2/2-(x^2\ln x)/2+x^2/4\right]_1^e=(e^2-1)/4$. Solving for what the question asks for: $58/15-1+\pi/4+e^{\ln(5/4)}+(e^2-1)/4-58/15=(e^2+\pi)/4$.

14. $(18+3\sqrt{3})$ $Area=(1/2)\cdot 12\cdot 12\cdot\sin\theta=72\sin\theta\rightarrow dArea=72\cos\theta d\theta=72(1/2)(-1/2)=-18=A$.
$Per.=24+12\sqrt{2(1-\cos\theta)}\rightarrow dPer.=12\sqrt{2}\sin\theta d\theta/\left(2\sqrt{1-\cos\theta}\right)=6\sqrt{2}\left(\sqrt{3}/2\right)(-1/2)/\sqrt{1/2}=-3\sqrt{3}=B$. So $|A|+|B|=18+3\sqrt{3}$.

15. $[(218/25)-\sqrt{2}]$ $f(\sqrt{3})=4\sqrt{3}/\sqrt{8(\sqrt{3})^2+1}=4\sqrt{3}/5\rightarrow\left[f(\sqrt{3})\right]^2=48/25$.
$f(x)=4x(8x^2+1)^{-1/2}$, so $f'(x)=4x(-1/2)(8x^2+1)^{-3/2}(16x)+4(8x^2+1)^{-1/2}\rightarrow f'(-\sqrt{3})=(-96/125)+(4/5)\rightarrow 25f'(-\sqrt{3})=4/5$
$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}1/\sqrt{(8x^2+1)/16x^2}=\lim_{x\to-\infty}1/\sqrt{(1/2)+(1/16x^2)}=\sqrt{2}$, but remember we're going to $-\infty$, so the answer is $-\sqrt{2}$.
$\int_0^{\sqrt{3}}4x\,dx=\left(2x^2\right]_0^{\sqrt{3}}=6$. Putting it all together gets $(218/25)-\sqrt{2}$.