1 Solution Key to Matrices, Determinants and Vectors Mu Level Topic Test

1999 Mu Alpha Theta Convention – Gatlinburg, TN

1. $3(33) - 5(25) = 99 - 125 = \underline{-26}$ C

2. $\begin{aligned}3x + 2y &= 4\\ 5x + 5y &= 5\end{aligned} \Rightarrow \begin{aligned}3x + 2y &= 4\\ x + y &= 1\end{aligned} \Rightarrow \begin{aligned}3x + 2y &= 4\\ -\ 2x + 2y &= 2\\ \hline x &= 2\\ y &= -1\end{aligned}$

$2 + y = 1 \Rightarrow$

$x + y =$
$2 + (-1) = \underline{1}$

C

3. $-13440 + 768x + 16200 - 9600 - 756x + 23040 = 16347$

$12x + 16200 = 16347$

$12x = 147$

$x = \frac{147}{12} = \frac{49}{4} \Rightarrow 49 + 4 = \underline{53}$

C

4. $A \cdot B = \begin{bmatrix}(-32+28+75) & (18-56-210)\\ (-192+40-20) & (108-80+56)\end{bmatrix} = \begin{bmatrix}71 & -248\\ -172 & 84\end{bmatrix}$ B

5. $3a - 2b + c = 3(i - 2j + 3k) - 2(4i + 3j - 5k) + (-2i + j + 3k)$

$= 3i - 6j + 9k - 8i - 6j + 10k - 2i + j + 3k$

$= \underline{-7i - 11j + 22k}$ B

6. $x = |u| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$

$y = |v| = \sqrt{1^2 + 4^2 + (-2)^2} = \sqrt{1 + 16 + 4} = \sqrt{21}$

$u \cdot v = (2)(1) + (-3)(4) + (1)(-2) = 2 - 12 - 2 = -12$

D

$x^2 + y^2 + z^2 = (\sqrt{14})^2 + (\sqrt{21})^2 + (-12)^2 = 14 + 21 + 144 = \underline{1}$

(7) $\cos\theta = \dfrac{a\cdot b}{|a||b|} = \dfrac{(3)(1)+(2)(-2)+(2)(-1)}{\sqrt{3^2+2^2+2^2}\cdot\sqrt{1^2+(-2)^2+(-1)^2}}$

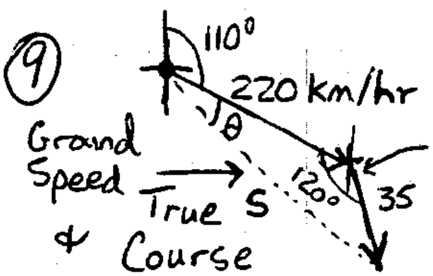
$$\cos\theta = \frac{3-4-2}{\sqrt{17}\cdot\sqrt{6}} = \frac{-3}{\sqrt{17}\cdot\sqrt{6}}$$

(A)

$\theta = \cos^{-1}\left(\dfrac{-3}{\sqrt{17}\sqrt{6}}\right)$ using calculator $\underline{\theta = 107.3}$

(8) Inverse (Using Calculator) $= \begin{bmatrix} -.6 & -.2 & .2 \\ .4 & .2 & .8 \\ .3 & -.1 & .6 \end{bmatrix} \Rightarrow$ Sum of these elements $= \underline{2.8}$ (C)

(9)



First, use law of cosines to find the ground speed of the plane

$$S^2 = 220^2 + 35^2 - 2(220)(35)\cos 120°$$
$$S^2 = 48400 + 1225 - (-7700)$$
$$S^2 = 57325$$
$$S = \sqrt{57325}$$

Second, use the law of sines to find θ.

$$\frac{\sin\theta}{35} = \frac{\sin 120}{\sqrt{57325}} \Rightarrow \sin\theta = \frac{35\sin 120}{\sqrt{57325}} \Rightarrow \theta = \sin^{-1}\left(\frac{35\sin 12}{\sqrt{57325}}\right)$$

$\theta \approx 7.273°$

Third, The true heading is $110° + \theta$

$110 + 7.273 \cong 117.273 \cong \underline{117°}$ (B)

(10) $([P]+[Q])^2 = [P]^2 + [P][Q] + [QP] + [Q]^2$

Since Matrix Multiplication is not commutative, Answer must remain in above form (E)

(11) A vector which is linearly dependent to u must be able to be written in the form k·u where k is a real number. None of the answer choices can be written in this form ∴ the answer is (E)

(12) $u \times v = \begin{vmatrix} i & j & k \\ 2 & -3 & 1 \\ 1 & 4 & -2 \end{vmatrix} = 6i + j + 8k + 3k - 4i + 4j$ (A)

$= 2i + 5j + 11k = \underline{2i + 5j + 11k}$

(13) let $\alpha = \angle$ that A makes i, $\beta = \angle$ A makes with j and $\gamma \angle A$ makes with k

Then by direction cosine theorem $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\alpha = 57^\circ \quad \gamma = 44^\circ \Rightarrow \cos^2 57^\circ + \cos^2\beta + \cos^2 44^\circ = 1$

$\cos^2\beta = 1 - \cos^2 57^\circ - \cos^2 44$

$\cos^2\beta \simeq .1859 \qquad \cos\beta \doteq .4312$ or $\cos\beta \doteq -.4312$

$\cos\beta \simeq \pm .4312 \qquad \beta \doteq 64^\circ \qquad \beta \simeq 115.54$

$\underline{\beta \simeq 116^\circ}$ (D)

Largest Angle

(14) The rank of a matrix is the order of the highest order non-zero determinant which can be formed by deleting rows and/or columns. $\therefore$ $\begin{vmatrix} 5 & -2 \\ 0 & -1 \end{vmatrix}$ has an order of 2 and therefore the rank of the matrix is $\underline{2}$. (A)

(15) $V = \frac{1}{6} \begin{vmatrix} 2 & 1 & -3 & 1 \\ 1 & 0 & 2 & 1 \\ 3 & -1 & 1 & 1 \\ 0 & 1 & 3 & 1 \end{vmatrix} \Rightarrow$ by calculator $= \frac{1}{6}|-6| = \frac{1}{6}(6) = \underline{1}$

(B)

(16) The trace of a matrix is the sum of the elements in the principal diagonal. $\therefore$ $5+2+4+3 = \underline{14}$ (C)

(17) Equation of a plane is $Ax + By + Cz + D = 0$

The coefficients A, B and C are equal to the respective i, j, k components of the normal. Thus we have: (C)

$3x + 2y - 5z + D = 0$ since $u = 3i + 2j - 5k$ is the normal

substituting $(1, 1, 2)$, we find D $\quad 3 + 2 - 10 + D = 0 \quad D = 5 \Rightarrow \underline{3x + 2y - 5z + 5}$

(18) The components of the direction vector come from the coefficients of the parameter (t) of the respective parametric equation.

$$ai + bj + ck \Rightarrow \begin{cases} x = 2 + at \\ y = 3 + bt \\ z = 1 + ct \end{cases}$$

(D)

$\therefore$ $\underline{i + j - 2k}$

(19) An adjoint matrix is the transpose of the cofactors of the matrix. $\therefore$ The adjoint of $A = \begin{bmatrix} -1 & 8 & -5 \\ 3 & 13 & 15 \\ 10 & -6 & 13 \end{bmatrix}$ Element in the 1st row 3rd Column

15 $\underline{-5}$ (B)

(20) $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Image Under $R_{135°} = \begin{bmatrix} \cos 135° & -\sin 135° \\ \sin 135° & \cos 135° \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

(D)

(21) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$

$2x + y = \lambda x \Rightarrow y = \lambda x - 2x$

$x + 2y = \lambda y$

Substituting Yields

$x + 2(\lambda x - 2x) = \lambda(\lambda x - 2x)$

$x + 2\lambda x - 4x = \lambda^2 x - 2\lambda x$

$\lambda^2 x - 4\lambda x + 3x = 0$

$x(\lambda^2 - 4\lambda + 3) = 0$

$x(\lambda - 3)(\lambda - 1) = 0$

$\lambda = 3 \quad \lambda = 1$ (B)

(22) $W = \vec{F} \cdot \vec{D}$ F = force, D = distance

$W_{A \to B} = \vec{F} \cdot \vec{AB} = (10i - 5j) \cdot (5i + 7j) = (10)(5) + (-5)(7) = 50 - 35 = 15\,J$

$W_{B \to C} = \vec{F} \cdot \vec{BC} = (10i - 5j) \cdot (2i - 7j) = (10)(2) + (-5)(-7) = 20 + 35 = 55$

Total Work $= 15\,J + 55\,J = \underline{70 \text{ Joules}}$ (D)

(23) $z = \dfrac{\begin{vmatrix} -3 & 4 & 3 \\ 1 & 2 & 9 \\ 0 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} -3 & 4 & -1 \\ 1 & 2 & -3 \\ 0 & 1 & -5 \end{vmatrix}} = \dfrac{6+0+3-0+27+4}{30+0-1+0-9+20} = \dfrac{40}{40} = \underline{1}$ C

(24) First, Find two points which lie on the intersection of the two planes:

From 1st Plane $x = 3 - y + z$

Sub into 2nd Plane $3 - y + z - y + 3z = 1$

$-2y + 4z = -2$

$-y + 2z = -1$

If $y = 1$ $z = 0$ $x = 2$

If $y = -1$ $z = -1$ $x = 3$

The two points chosen are

$(2, 1, 0)$

$(3, -1, -1)$

Second, Use $r = (1-t)r_0 + t r_1$ where r_0 and r_1 are position vectors of the two chosen points

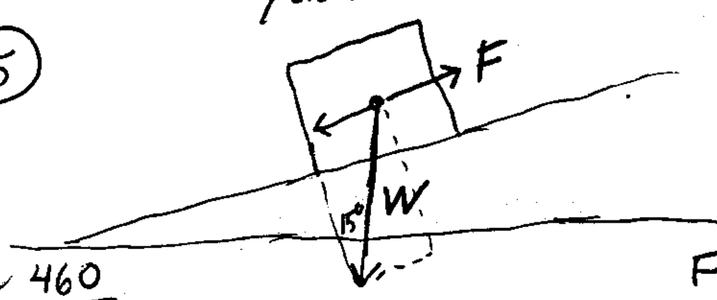
$r = (1-t)(2i + j) + t(3i - j - k)$

$r = 2i + j - 2ti - tj + 3ti - tj - tk$

$r = (2+t)i + (1-2t)j - tk$ D

Note: If two other points were chosen, you must equate parameters and adjust to see if the other equation matches D.

(25)



Forces Acting on Crate to move down

$+20\text{ N} + W(\sin 15^\circ) - F$ C

Forces Acting on Crate to Move Up

$+480\text{N} - W(\sin 15^\circ) - F$

Equating Forces yields

$20 + W\sin 15 - F = 480 - W\sin 15 - F$

$2W\sin 15 = 460$

$W = \dfrac{460}{2\sin 15}$

$W = 9.8m$

$\dfrac{9.8m}{9.8} = \dfrac{460}{9.8(2\sin 15)}$

$m \approx 90.17 = \boxed{91\text{Kg}}$

F = Frictional Force of Ramp

(26) Let α = $\angle$ the vector makes with i, j ank

Using to direction cosine theorem $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

we have, $\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

$$3\cos^2\alpha = 1$$

$$\cos^2\alpha = \frac{1}{3}$$

$\cos\alpha = \pm\frac{1}{\sqrt{3}}$

Acute $\angle$ (C)

$\cos\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.7$

(27) $\cos A \cos B - \sin A \sin B$

$\cos(A+B)$ (A)

(28) A Jordan block is a matrix in which all elements in the principal diagonal are equal (and non-zero), all elements in the superdiagonal are 1 and all other elements are 0. (D)

(29) $\boxed{\text{Method 1}}$

Use dot product

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$$

Let $\vec{A}$ & $\vec{B}$ represent the normal vectors of each plane respectively

$\vec{A} = 2i - j + 3k$

$\vec{B} = i - 2j + k$

$$\cos\theta = \frac{2(1)+(-1)(-2)+(3)(1)}{\sqrt{2^2+(-1)^2+3^2}\sqrt{1^2+(-2)^2+1^2}}$$

$$\cos\theta = \frac{7}{\sqrt{14}\sqrt{6}} =$$

$$\theta = \cos^{-1}\left(\frac{7}{\sqrt{84}}\right) = 40.2^0$$

$\boxed{\text{Method 2}}$

Use x product

$\vec{A}\times\vec{B} = 5i + j - 3k$

$$\sin\theta = \frac{|\vec{A}\times\vec{B}|}{|\vec{A}||\vec{B}|} = \frac{\sqrt{5^2+1^2+(-3)^2}}{\sqrt{14}\sqrt{6}}$$

$$\sin\theta = \frac{\sqrt{35}}{\sqrt{14}\sqrt{6}}$$

$$\theta = \sin^{-1}\left(\frac{\sqrt{35}}{\sqrt{84}}\right)$$

$$\theta = 40.2^0$$

(A)

(30) $f'(x) = \frac{(x-2)(1) - x(1)}{(x-2)^2}$

$f'(x) = \frac{-2}{(x-2)^2}$

Slope at $(4,2)$ $= \frac{-2}{(4-2)^2} = \frac{-2}{4} = \frac{-1}{2}$

Vector tangent a $(4,2)$ must have a slope of $-\frac{1}{2}$.

$2i - j$ is such a vector

$\therefore \frac{2i - j}{\sqrt{1^2+(-2)^2}} = \frac{2}{\sqrt{5}}i - \frac{1}{\sqrt{5}}j$ (A)

$A + B = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

TIE BREAKER--MATRICES & VECTORS
SOLUTIONS

1. Find $\operatorname{proj}_{\bar{u}} \bar{v}$ given $\bar{u} = 3\bar{i} - 5\bar{j}$ and $\bar{v} = 6\bar{i} + 2\bar{j}$.

$$\bar{v} \bullet \bar{u} = 6*3 + 2*-5 = 8$$

$$\|\bar{u}\|^2 = 3^2 + (-5)^2 = 34$$

$$\operatorname{proj}_{\bar{u}} \bar{v} = \left(\frac{\bar{v} \bullet \bar{u}}{\|\bar{u}\|^2}\right)\bar{u}$$

$$\operatorname{proj}_{\bar{u}} \bar{v} = \frac{8}{34}\left(3\bar{i} - 5\bar{j}\right) = \tfrac{12}{17}\bar{i} - \tfrac{20}{17}\bar{j}$$

2. Solve the following equation for all values of x.

$$\begin{vmatrix} 1-x & 1 & -1 \\ 2 & 3-x & -4 \\ 4 & 1 & -4-x \end{vmatrix} = 0.$$

$$(1-x)\begin{vmatrix} 3-x & -4 \\ 1 & -4-x \end{vmatrix} - 1\begin{vmatrix} 2 & -4 \\ 4 & -4-x \end{vmatrix} + -1\begin{vmatrix} 2 & 3-x \\ 4 & 1 \end{vmatrix} = 0$$

$$(1-x)(-8 + x + x^2) - 1(8-2x) - 1(-10 + 4x) = 0$$

$$-6 + 7x - x^3 = 0 \Rightarrow x^3 - 7x + 6 = 0$$

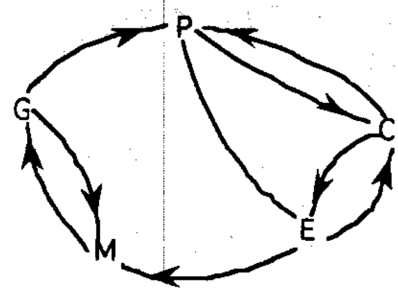
$$\begin{array}{r|rrr|r} & 1 & 0 & -7 & 6 \\ \hline 1 & 1 & 1 & -6 & 0 \end{array}$$

$$(x-1)(x^2 + x - 6) = 0$$

$$(x-1)(x+3)(x-2) = 0$$

$$x = 1, 2, -3$$

3. The entrance of Euler National Park is connected to four other sites in the park by rather primitive roads. From the entrance of Euler Park, a tourist can drive his car to either Cramer Cove or Mandelbrot Meadow. From Mandelbrot Meadow, he can drive to Gauss Gorge. From Gauss Gorge, he can drive back to Mandelbrot Meadow or on to Pascal Peak. From Pascal Peak he could drive to Cramer Cove or back to Euler Entrance. From Cramer Cove he could drive back to Euler Entrance or to Pascal Peak. From Pascal Peak, how many ways can he get back to Euler Entrance going through at most one other site?



$$A = \begin{array}{c} \\ E \\ C \\ M \\ G \\ P \end{array} \begin{array}{c} \begin{array}{ccccc} E & C & M & G & P \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \qquad A^2 + A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 0 & 1 \end{bmatrix}$$

There are 2 ways from Pascal Peak to Euler Entrance that the tourist could take and go through <u>at most</u> one other site.