1999 Mu Alpha Theta National Convention
Alpha Equations & Inequalities Solutions

NOTA = none of these answers

1. The degree of $(x^2 + 1)^4 (x^3 + 1)^3$ as a polynomial in x is what?

Answer: D
Solution: The highest powers of x in the factors $(x^2 + 1)^4$ and $(x^3 + 1)^3$ are 8 and 9, respectively. Therefore, the highest power of x in the product is 8 + 9 = 17.

2. The equation $x^6 - 3x^5 - 6x^3 - x + 8 = 0$ has

Answer: D
Solution: For $x < 0$, the polynomial $x^6 - 3x^5 - 6x^3 - x + 8$ is positive, since then all terms are positive; so it has no negative zeros. At x = 1, the polynomial is negative and hence has at least one positive zero (between 0 and 1).

3. The smallest root of the equation $(x - 0.75)(x - 0.75) + (x - 0.75)(x - 0.5) = 0$ is:

Ans: C 5/8
Solution: By factoring you get $(x - .75)(2x - 1.25) = 0$ $x = .75$ or $x = 5/8$

4. A college graduate goes to work for \$x per week. After several months the company falls on hard times and gives all the employees a 10% pay cut. A few months later, business picks up and the company gives all the employees a 10% raise. What is the college graduate's new salary?

Ans: B
Solution: After the pay cut the graduate was making \$0.90x. Then with the pay raise of 10%, 1.1 (.90x) = .99x

5. What is the nature of the roots of the equation $ax^2 + bx + c = 0$, $a \neq 0$, and a and c have different signs.

Ans: B
Solution: If a and c have different signs, then $ac < 0$. Thus, $-4ac > 0$, $b^2 - 4ac$ would have a positive value indicating two real roots.

6. Determine the number of integral value(s) of k such that $x^3 - 4x^2 - 2x + k = 0$ has a root between - 1 and - 2.

Ans: E $3 < k < 20$
Solution: Substitute the values of - 1 and - 2 into the equation for x and solve for k.

7. The discriminant of $(2 + \sqrt{3})x^2 - (4 - \sqrt{3})x = 1$ is what?

Ans: D
Solution: The discriminant is $b^2 - 4ac$. So,

$$(4 - \sqrt{3})^2 - 4(2 + \sqrt{3})(-1) = 16 - 8\sqrt{3} + 3 + 8 + 4\sqrt{3} = 27 - 4\sqrt{3}$$

8. If $\frac{k-3x}{m-2x} = 1$, then what does x equal?

Ans: A
Solution: $\frac{k-3x}{m-2x} = 1$, k - 3x = m - 2x, x = k - m

9. Solve for x: $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \sqrt{15} + x$

Ans: B
Solution: Rationalizing the left hand side the equation becomes $\frac{(\sqrt{5}+\sqrt{3})^2}{2} = \sqrt{15} + x$, simplifying and solving for x you get x = 4.

10. A swimming pool has two inlet pipes, each of which alone can fill the pool in 3 hours and 4 hours respectively. A third outlet pipe alone can empty the pool in 1.5 hours. If the pool is only half filled and then all 3 pipes are opened simultaneously, how long will it take to empty or fill the pool?

Ans: D
Solution: $\frac{x}{3} + \frac{x}{4} - \frac{2x}{3} = \frac{1}{2}$ Solving for x, you get x = - 6 which means that it takes 6 hours to empty the ½ of the pool.

11. X varies directly as R^2 and inversely as S. If R is multiplied by 2, by what number must S be multiplied in order the X remains unchanged?

Ans: D
Solution: $\frac{SX}{R^2}$ $\frac{SX}{(2R)^2} = \frac{SX}{4R^2}$ So S needs to be multiplied by 4, to keep X unchanged.

12. Solve the following system of inequalities: 4 + r < - 3 and - 2 + r ≥ - 9

Ans: E
Solution: Solving the pair of inequalities, r < - 7 and r ≥ - 7. Therefore, the intersection of the two inequalities is the empty set.

13. If x is a real number, then the quantity (1 - | x |)(1 + x) is positive if and only if

Answer: E
Solution: The quantity (1 - | x |)(1 + x) is positive if and only if either both factors are positive or both factors are negative. Both factors are positive if and only if - 1 < x < 1, while both factors are negative if and only if x < - 1. Therefore, x < - 1 or - 1 < x < 1

14. The product of the roots of the quadratic equation $x^2 - 3kx + 2k^2 - 1 = 0$ is equal to 7. The roots may be characterized as:
Ans: D
Solution: The product of the roots of a quadratic equation $ax^2 + bx + c = 0$ is c/a. Therefore, since the product of the roots of the given equation is equal to 7 you get $2k^2 - 1 = 7$; k = ± 2. Substituting k = ± 2 back into the equation you get $x^2 - 6x + 7 = 0$ or $x^2 + 6x + 7 = 0$ Solving for x you get $3 \pm \sqrt{2}$ or $-3 \pm \sqrt{2}$ which are irrational.

15. If $-2 < \frac{1}{x} < 3$, then :

Ans: C
Solution: Solving $-2 < \frac{1}{x} < 3$; $-2x < 1 < 3x$; $x > \frac{-1}{2}$ and $x > \frac{1}{3}$; Therefore, $x > \frac{1}{3}$

16. If $\frac{1}{x} - \frac{1}{y} = \frac{1}{z}$ then z equals what?

Ans: B

Solution: $\frac{1}{x} - \frac{1}{y} = \frac{1}{z}$; $\frac{y-x}{xy} = \frac{1}{z}$; $\frac{xy}{y-x} = z$

17. The set of all real solutions of the inequality $|x - 1| + |x + 2| < 5$

Answer: A
Solution: Checking the 4 possible inequality statements: 1) $x - 1 + x + 2 < 5$; $x < 2$
2) $x - 1 - x + 2 < 5$; $1 < 5$ 3) $-x + 1 + x + 2 < 5$; $3 < 5$ 4) $-x + 1 - x - 2 < 5$; $x > -3$
The final solution therefore is: $-3 < x < 2$

18. Solve $\frac{a^x - a^{-x}}{2} = 3$ for x where $a > 0$.

Ans: B

Solution: $\frac{a^x - a^{-x}}{2} = 3$; $a^x - a^{-x} = 6$; $a^{2x} - 1 - 6a^x = 0$; $a^{2x} - 6a^x - 1 = 0$; let $y = a^x$,
then $y^2 - 6y - 1 = 0$ $y = 3 \pm \sqrt{10}$; $a^x = 3 \pm \sqrt{10}$; $x = \text{Log}_a(3 + \sqrt{10})$ other solution not possible

19. Let x = Cos 36° - Cos 72°. Then x equals what?

Answer: D
Solution: Let x = Cos 36° and y = Cos 72°. Applying the identities $\text{Cos}2\theta = 2\text{Cos}^2\theta - 1$ and $\text{Cos } 2\theta = 1 - 2\text{Sin}^2\theta$ with θ = 36° in the first identity and θ = 18° in the second identity yields
$y = 2w^2 - 1$ and $w = 1 - 2y^2$ Adding these last two equations yields
$w + y = 2(w^2 - y^2) = 2(w - y)(w + y)$ and division by $w + y$ yields $2(w - y) = 1$, so $x = w - y = ½$

20. The difference of two positive numbers is 2, and the product of these two numbers is 17. What is their sum?

Answer: A
Solution: If x and y denotes the two numbers, then $(x + y)^2 = (x - y)^2 + 4xy = 2^2 + 4(17) = 72$
Therefore, $x + y = 6\sqrt{2}$

21. One solution of $x^3 + 5x^2 - 2x - 4 = 0$ is $x = 1$. Which of the following is another solution?

Answer: B
Solution: Since x = 1 is a root then it is a factor. $x^3 + 5x^2 - 2x - 4 = 0$ $(x - 1)(x^2 + 6x + 4) = 0$ By using the quadratic formula the other solutions are $-3 + \sqrt{5}$, $-3 - \sqrt{5}$

22. Herman's car gets 3 more miles per gallon during highway driving than it does during city driving. On a recent trip, Herman drove 136 miles on the highway and 155 miles in the city, using a total of 9 gallons of gasoline. How many miles per gallon does Herman's car get during city driving?

Answer: D
Solution: Let x denote the number of miles per gallon Herman gets during city driving so that

x + 3 is the number of miles per gallon Herman get during highway driving. $\frac{155}{x} + \frac{136}{x+3} = 9$;
$9x^2 - 264x - 465 = 0$ Factoring, $3(3x + 5)(x - 31) = 0$, so that x = 31

23. John and Nancy live on the same street and often walk towards each other's home. If they both leave their home at 8:00 a.m., then they will meet at 8:04 a.m. If Nancy leaves her home at 8:00 a.m. but John does not leave his home until 8:03 a.m., then they will meet at 8:05 a.m. How many minutes does it take John to walk all the way to Nancy's home? Assume that each person walks at his or her own constant rate.

Answer: E
Solution: Let x be John's walking speed and y be Nancy's walking speed both in feet per minute, and let d denote the distance in feet between the houses. Then in 4 minutes, John travels 4x feet and Nancy travels 4y feet so that d = 4x + 4y. Similarly, one gets d = 2x + 5y. Solving for d in terms of x you get d = 12x. This implies it will take John 12 minutes to travel distance d.

24. Which point below is in the interior of the circle $x^2 + 6y + y^2 = 7$?

Answer: D
Solution: Check the points in the inequality $x^2 + 6y + y^2 < 7$; A) $9 - 36 + 36 \not< 7 \quad 9 \not< 7$
B) $1 - 42 + 49 \not< 7 \quad 8 \not< 7$ C) $16 - 12 + 4 \not< 7 \quad 8 \not< 7$ D) $9 - 6 + 1 < 7 \quad 4 < 7$ **

25) Determine the equation of the circle whose center is (4, - 4) and whose radius is the sum of

the following geometric series: $3\frac{1}{2} + 1\frac{3}{4} + \frac{7}{8} + \ldots$

Answer: C
Solution: $r = \frac{7}{2} + \frac{7}{4} + \frac{7}{8} + \ldots = 7(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots) = 7(1) = 7$ The equation of the circle is

$(x - 4)^2 + (y + 4)^2 = 49$

26. If $\frac{n!}{2} = (n-2)!$ then n = ?

Answer: B
Solution: $\frac{n!}{2} = (n-2)!$ $\quad n(n - 1) = 2 \quad n^2 - n - 2 = 0 \quad (n - 2)(n + 1) = 0 \quad n = 2, n = -1$ Only

possible answer is when n = 2

27. If log x = ½ log a - log b and a = $4b^2$, find x.

Answer: B
Solution: $\log x = \log \frac{a^{\frac{1}{2}}}{b} \qquad x = \frac{a^{\frac{1}{2}}}{b} \qquad \frac{(4b^2)^{\frac{1}{2}}}{b} = \frac{2b}{b} = 2$

28. If the perpendicular bisector of the segment with endpoints A(1,2) and B(2,4) contains the point (4,c) what is the value of c?

Answer: C
Solution: (4, c) must be equidistant from A and B so $\sqrt{(4-1)^2+(c-2)^2} = \sqrt{(4-2)^2+(c-4)^2}$

$c^2 - 4c + 13 = c^2 - 8c + 20 \qquad c = \frac{7}{4}$

29. Make a sketch of the graphs of $y = |x+2|$ and $y = -|x| + 2$ and determine the number of points in the intersection. How many are there?

Answer: D
Solution: Sketch the picture and you'll see a line segment where the graphs overlap and contains infinitely many points.

30. If $y = |\text{Sin } x|$ then for all x the range of values of y is:

Answer: B
Solution: Sin x has a maximum value of 1 and a minimum value of - 1. Thus the absolute value of sin x varies from 0 to 1.

Tiebreaker Questions

T1. Given that $x^6 + 4x^5 + 6x^4 + 6x^3 + 4x^2 + 2x + 1$ can be factored as $(x^2 + ax + 1)(x^4 + bx^3 + cx^2 + dx + 1)$, what is the value of $a + b$?

Answer: 4
Solution: Since $(x^2 + ax + 1)(x^4 + bx^3 + cx^2 + dx + 1) = a^6 + (a+b)x^5 + \ldots + 1$, we see that a+b is the coefficient of x^5, which is 4.

T2. For which real values of m are the simultaneous equations $y = mx + 3$; $y = (2m - 1)x + 4$ satisfied by at least one pair of real numbers (x, y) ?

Answer: all $m \neq 1$
Solution: The equations have a solution unless their graphs are parallel lines. This will be the case only if the slopes are equal, i.e. if $m = 2m - 1$ or $m = 1$ (The lines are not coincident since they have distinct y-intercepts for all values of m.)

T3. Find a positive integral solution to the equation: $\frac{1+3+5+\ldots+(2n-1)}{2+4+6+\ldots+2n} = \frac{115}{116}$

Answer: 115
Solution: Summing the arithmetic progressions yields

$$\frac{115}{116} = \frac{1+3+5+\ldots+(2n-1)}{2+4+6+\ldots+2n} = \frac{n^2}{n(n+1)} = \frac{n}{n+1} \qquad n = 115$$