Limits and Derivatives - Solutions 1999 MAΘ Convention

① $\dfrac{|3(-2)-5|}{5(-2)-3} = \dfrac{|-11|}{-13} = -\dfrac{11}{13}$ Ⓐ

② $\lim\limits_{n\to\infty}\left[2-\dfrac{1}{n}\right]^5 = \left[2-0\right]^5 = 2^5 = 32$ Ⓓ

③ $\dfrac{dy}{dx} = (3x-2)^{-1} = -1(3x-2)^{-2}(3) = -3(3x-2)^{-2}$

$\dfrac{d^2y}{dx^2} = 6(3x-2)^{-3}(3) = 18(3x-2)^{-3} = \dfrac{18}{(3x-2)^3}$ Ⓓ

④ $\lim\limits_{x\to\infty}\left(\dfrac{2x^2-x}{x^2-4}\right)^{\frac{1}{x}} = 2^0 = 1$ Ⓒ

⑤ $\dfrac{a}{1-r} = \dfrac{500}{1-\frac{1}{5}} = \dfrac{500}{\frac{4}{5}} = 625$ Ⓓ

⑥ $f^{-1}(x) \Rightarrow \quad y = x^{3/4} \Rightarrow x = y^{3/4} \Rightarrow y = x^{4/3}$

$f^{-1}(x) = y = x^{4/3} \quad y' = \frac{4}{3}x^{\frac{1}{3}}$ at $x=27$ $y' = \frac{4}{3}(27)^{\frac{1}{3}} = 4$ Ⓒ

⑦ when $x = \frac{3}{2}$ $f(x) = \frac{0}{0}$ L'Hospital's Rule $\dfrac{24x^2}{4x+7}$ at $x = \frac{3}{2}$

gives $L = \dfrac{24\left(\frac{9}{4}\right)}{4\left(\frac{3}{2}\right)+7} = \dfrac{54}{13}$ Ⓒ

⑧ A) $L = 0$ b/c $\frac{0}{\cos 0} = \frac{0}{1} = 0$

B) $L = 0$ b/c sandwich theorem ∴ Since all limits are $= 0$ NOTA ✓

C) $L = 0$ b/c L'Hospital's Rule $\frac{1}{x} \Rightarrow L = 0$ Ⓔ

D) $L = 0$ b/c Geometric with $|r| < 1$

(9) The given lim is the definition of $f'(x)$ for $f(x) = 5x^3 - 2x$
Therefore, $f'(x) = 15x^2 - 2$ (B)

(10) $0.41 < \dfrac{2n+11}{5n} < .42$

$41n < 40n + 220 < 42n$ Multiplied by 100n

$41n < 40n + 220$ and $40n + 220 < 42n$

$n < 220$ and $-2n < -220$

$n > 110$

$n \in \langle 110, 220 \rangle$ (A)

$220 - 110 = 110 - 1 = \boxed{109}$

220 must be excluded

(11) $f'(x) = 3x^2 + 6x - 9 = 0$

$3(x^2 + 2x - 3) = 0$

$(x+3)(x-1) = 0$

$x = -3$ $\quad$ $x = 1$

$(-3, 5)$ $\quad$ $(1, -27)$

Rel. Max. $\quad$ Rel Min

$f(x)$ values

	1	3	-9	-22
-3	1	0	-9	5
1	1	4	-5	-27
-1	1	2	-11	-11

$f''(x) = 6x + 6 = 0$

$x = -1$

$(-1, -11)$

$a = -3$
$b = 5$
$c = 1$
$d = -27$
$e = -1$
$f = -11$

$3a + 3b + 2c + 2d + e + f =$

$3(2) + 2(-26) - 12 = 0$

$6 - 52 - 12 = -58$

(B)

(12) Use the product rule

$h'(x) = f'(x)g(x) + f(x)g'(x)$

$= \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}$

$= \dfrac{x \cdot \cos x \cdot \ln x + \sin x}{x}$ (C)

(13) $$\lim_{x \to 0} \frac{\tan 7x}{\sin 2x} = \lim_{x \to 0}\left[\left(\frac{1}{\cos 7x}\right)\left(\frac{\sin 7x}{1}\right)\left(\frac{1}{\sin 2x}\right)\right] \Rightarrow$$

$$= \lim_{x \to 0}\left[\left(\frac{1}{\cos 7x}\right)\left(\frac{\sin 7x}{7x}\right)\left(\frac{7x}{1}\right)\left(\frac{2x}{\sin 2x}\right)\left(\frac{1}{2x}\right)\right] = (1)(1)(1)\left(\frac{7}{2}\right) = \frac{7}{2}$$

(B)

(14) $$f'(x) = \frac{(x-1)\frac{1}{2}(x+1)^{-\frac{1}{2}} - \sqrt{x+1}\,(1)}{(x-1)^2} = \frac{x-1}{2(x-1)^2\sqrt{x+1}} - \frac{\sqrt{x+1}}{(x-1)^2} \Rightarrow$$

Using Quotient Rule

$$= \frac{x-1 - \left[\sqrt{x+1}\cdot 2 \cdot \sqrt{x+1}\right]}{2(x-1)^2\sqrt{x+1}} = \frac{x-1-[2x+2]}{2(x-1)^2\sqrt{x+1}} = \frac{-x-3}{2(x-1)^2\sqrt{x+1}}$$

(B)

(15) $$\lim_{n \to 0^+} \frac{\sqrt{n}}{\sqrt{9+\sqrt{n}}-3} \cdot \frac{\sqrt{9+\sqrt{n}}+3}{\sqrt{9+\sqrt{n}}+3} = \lim_{n \to 0^+} \frac{\sqrt{n}\left(\sqrt{9+\sqrt{n}}+3\right)}{9+\sqrt{n}-9}$$

$$= \lim_{n \to 0^+} \frac{\cancel{\sqrt{n}}\left(\sqrt{9+\sqrt{n}}+3\right)}{\cancel{\sqrt{n}}} = \sqrt{9+\sqrt{0}}+3 = 3+3 = \underline{\underline{6}}$$

(D)

(16) Find $f'(-2)$: $$f'(x) = \frac{(x^2-9)(2) - (2x+1)(2x)}{(x^2-9)^2}$$

$$f'(x) = \frac{-2x^2-2x-18}{(x^2-9)^2} \Rightarrow f'(-2) = \frac{-2(4)+4-18}{(-5)^2}$$

$$f'(-2) = \frac{-22}{25}$$

$$|A+B| = |-22+25| = \underline{\underline{3}}$$

(A)

(17) I. TRUE at $x=4$ $f(x) = x+3 = 4+3 = 7$ ✓

II. TRUE as $x \to 4^-$ $L = 8$ and as $x \to 4^+$ $L=8$

III. FALSE $f(4) \neq \lim_{x \to 4} f(x)$ $\boxed{\therefore L = 8}$ ✓

IV. FALSE $f'(x)$ does not exist at discontinuity

I and II are True (D)

(18) $f(x) = \dfrac{g(x)}{h(x)} - \dfrac{h(x)}{g(x)}$

$$f'(0) = \frac{h(0)g'(0) - h'(0)g(0)}{[h(0)]^2} - \frac{g(0)h'(0) - g'(0)h(0)}{[g(0)]^2}$$

$$= \frac{(1)(3) - 4(5)}{1^2} - \frac{5(4) - 3(1)}{5^2}$$

$$\frac{-17}{1} - \frac{17}{25} = \frac{-17(25) - 17}{25} = \frac{-442}{25}$$

(C)

(19) $\displaystyle\lim_{h \to 0} \frac{2^{3+h} - 8}{h} = \lim_{h \to 0} \frac{2^{3+h} - 2^3}{h} = f'(3)$ for $f(x) = 2^x$

If $f(x) = 2^x$, then $f'(x) = 2^x \ln 2$, $f'(3) = \underline{8 \ln 2}$

(D)

(20) $F(x) = f(g(x))$ Use Chain Rule

$F'(x) = f'(g(x)) \cdot g'(x)$

$F'(5) = f'(g(5)) \cdot g'(5)$

$f'(12) \cdot g'(5)$

$9 \cdot 7 = \underline{63}$ (C)

(21) $x^2 + xy - y^2 = 1$ Using Implicit Differentiation

$$2x + x\frac{dy}{dx} + y - 2y\frac{dx}{dx} = 0$$

at $(2,3)$ $2(2) + 2\frac{dy}{dx} + 3 - 6\frac{dy}{dx} = 0$

$$-4\frac{dy}{dx} = -7$$

$$\frac{dy}{dx} = \frac{7}{4}$$

Normal's Slope $= -\frac{4}{7}$

$$y - 3 = \frac{-4}{7}(x-2)$$

$$y = \frac{-4}{7}x + \frac{8}{7} + \frac{21}{7}$$

$$y = \frac{-4}{7}x + \frac{29}{7}$$

A

(22) $f'(x) > 0$ $f(x)$ is increasing
$f''(x) < 0$ $f(x)$ is concave down

These two conditions are met by curve in A

(23) $$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} \Rightarrow \tan t = \frac{\frac{d(\cos t)}{dt}}{\frac{dx}{dt}}$$

$$\frac{dx}{dt} = \frac{-\sin t}{\tan t} = -\cos t$$

D

(24) $$\lim_{x \to 0} (1+2x)^{\frac{1}{4x}} = \lim_{x \to 0} \left[(1+2x)^{\frac{1}{2x}}\right]^{\frac{1}{2}}$$

$$= e^{\frac{1}{2}} \quad \frac{A}{B} = \frac{1}{2} \quad 1 - 2 = -1$$

A

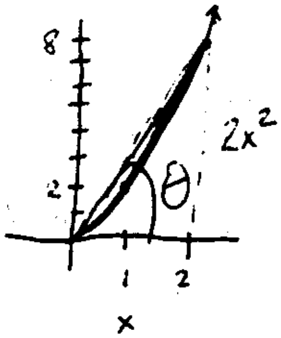
(25) $$\lim_{(x,y) \to \left(\frac{3\pi}{2}, B\right)} \cos(x+y) = \cos\frac{3\pi}{2}\cos B - \sin\frac{3\pi}{2}\sin B$$

$$0 - (-1)\sin B$$

$$= \sin B$$

B

(26)



$\tan\theta = \frac{2x^2}{x}$

$\tan\theta = 2x$

$\sec^2\theta \, \frac{d\theta}{dt} = 2\frac{dx}{dt}$

$\frac{d\theta}{dt} = \frac{2}{\sec^2\theta}\frac{dx}{dt} = \frac{2}{\quad}\left(\frac{6m}{sec}\right)$

$\frac{d\theta}{dt} = \frac{2}{(\sqrt{17})^2} \cdot 6 \ \frac{rad}{sec} = \frac{12}{17}\ \frac{rad}{sec}$

at $x = 2m$

$\tan\theta = \frac{4}{1}$

$\sec\theta = \frac{1}{\frac{1}{\sqrt{17}}} = \sqrt{17}$

(D)

(27) $y = \sqrt{x + \sqrt{1+\sqrt{x}}}$

$\frac{dy}{dx} = \frac{1}{2}\left(x + \sqrt{1+\sqrt{x}}\right)^{-\frac{1}{2}}\left(1 + \boxed{\frac{1}{2}}(1+\sqrt{x})^{-\frac{1}{2}}\left(\frac{1}{2}\right)\left(x^{-\frac{1}{2}}\right)\right)$

at $x = 9$ $\quad \frac{1}{2}\left(9 + \sqrt{1+\sqrt{9}}\right)^{-\frac{1}{2}}\left(1 + \frac{1}{2}(1+\sqrt{9})^{-\frac{1}{2}}\frac{1}{2}\,9^{-\frac{1}{2}}\right)$

$\frac{dy}{dx} = \frac{1}{2\sqrt{11}} \cdot \left(1 + \left(\frac{1}{2\cdot 2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\right)$

$= \frac{1}{2\sqrt{11}}\left(1 + \frac{1}{24} = \frac{25}{48\sqrt{11}} = \frac{25\sqrt{11}}{528}\right.$ (C)

(28) $\lim_{x\to\infty} \sqrt{x^2+x} - x = \lim_{x\to\infty} \frac{\sqrt{x^2+x} - x}{1} \cdot \frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}$

$= \lim_{x\to\infty} \frac{x^2+x-x^2}{\sqrt{x^2+x}+x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1}$

(E) $= \frac{1}{1+1} = \boxed{\frac{1}{2}}$ ✓

(29) $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 4 - \frac{f(4)}{f'(4)}$$

$$f(4) = \begin{array}{r|rrrrr} 4 & 1 & -8 & 1 & 66 & -46 \\ & & 4 & -16 & -60 & 24 \\ \hline & 1 & -4 & -15 & 6 & -22 \end{array} = \boxed{-22}$$

$$f'(x) = 4x^3 - 24x^2 + 2x + 66$$

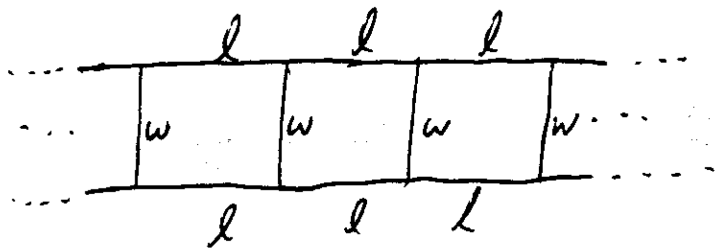
$$f'(4) = \begin{array}{r|rrrr} 4 & 4 & -24 & 2 & 66 \\ & & 16 & -32 & -120 \\ \hline & 4 & -8 & -30 & -54 \end{array}$$

$$x_{n+1} = 4 - \frac{(-22)}{(-54)}$$

$$= 4 - \frac{11}{27} \doteq 3.\underline{592}$$

$5+9+2 = \underline{16}$ (A)

(30)



l = length East to west of a pen

w = width North-South of pen

Let N = # of Pens

Perimeter $P = 2N(l) + (N+1)(w) = 600$

$$w = \frac{600 - 2Nl}{N+1}$$

$$A = Nlw$$

$$A = Nl\left(\frac{600 - 2Nl}{N+1}\right)$$

$$A = \frac{600Nl - 2N^2l^2}{N+1}$$

$$\frac{dA}{dl} = \frac{600N - 4N^2l}{N+1} = 0$$

$$600N = 4N^2l$$

$$l = \frac{600N}{4N^2}$$

$$l = \underline{\frac{150}{N}}$$

(C)

Tiebreakers

T1. Given f(x) = Arcsinx, g(x) = $\sqrt{x}$, h(x) = 1 - x^2, k(x) = 3x and $\phi = f(g(h(k(x))))$. Find $\phi'(\frac{1}{6})$

Answer: $-2\sqrt{3}$

Solution: $\phi(x) = Arcsin\sqrt{1-9x^2}$ $\quad \phi'(x) = \dfrac{\frac{1}{2}(1-9x^2)^{-\frac{1}{2}}(-18x)}{\sqrt{1-(1-9x^2)}} = \dfrac{-9x^2}{3x\sqrt{1-9x^2}} = \dfrac{-3}{\sqrt{1-9x^2}}$

$\phi'(\frac{1}{6}) = -2\sqrt{3}$

T2. Find the values of x for which $y = x^4 - 2x^3 - 12x^2$ is concave upward.

Answer: x < -1 or x > 2

Solution: $y' = 4x^3 - 6x^2 - 24x$ $\quad y'' = 12x^2 - 12x - 24$ $\quad y'' = 12(x-2)(x+1)$ Therefore, x < -1 or x > 2

T3. Find the limit: $\lim\limits_{x\to 2}\dfrac{4-x^2}{3-\sqrt{x^2+5}}$

Answer: 6

Solution: $\lim\limits_{x\to 2}\dfrac{(4-x^2)(3+\sqrt{x^2+5})}{(3-\sqrt{x^2+5})(3+\sqrt{x^2+5})} = \lim\limits_{x\to 2}\dfrac{(4-x^2)(3+\sqrt{x^2+5})}{4-x^2} = \lim\limits_{x\to 2}(3+\sqrt{x^2+5}) = 6$