Mu Alpha Theta National Convention 2004

Mu Integration

Answers

#	Answer	#	Answer
1	D	18	E
2	D	19	E
3	B	20	B
4	D	21	D
5	A	22	E
6	C	23	A
7	C	24	D
8	D	25	B
9	A	26	A
10	A	27	C
11	B	28	C
12	E	29	A
13	A	30	E
14	C	TB1	.79
15	D	TB2	$\frac{x^3}{3}\ln x - \frac{x^3}{9} + C$
16	C	TB3	$\frac{3}{\pi}$
17	D		

1. (D). $\int_1^2 (3x^3 - 7x^2 + 9)dx = \frac{3}{4}x^4 - \frac{7}{3}x^3 + 9x \Big|_1^2 = \frac{47}{12}$

2. (D). $\int_0^1 (x^5 - 2x^2 + 8x - 17)dx = \frac{1}{6}x^6 - \frac{2}{3}x^3 + 4x^2 - 17x \Big|_0^1 = \frac{-27}{2}$ The region in question is actually completely below the x-axis, so the absolute value of the above integral is the desired area.

3. (B). $\int_{\pi/6}^{\pi/4} (\sin x - \cos x)dx = -\cos x - \sin x \Big|_{\pi/6}^{\pi/4} = \frac{\sqrt{3}}{2} - \sqrt{2} + \frac{1}{2}$

4. (D). $\frac{1}{2 - \frac{1}{2}} \int_{1/2}^{2} \left(1 + 2x + \frac{1}{x^2}\right)dx = \frac{2}{3}\left(x + x^2 - \frac{1}{x}\right)\Big|_{1/2}^{2} = \frac{9}{2}$

5. (A). $\int_1^{\ln 2} \left(\frac{1}{x} + e^x\right)dx = \ln x + e^x \Big|_1^{\ln 2} = \ln(\ln 2) + 2 - e$

6. (C). Set y-values equal to each other to find intersection values y = -2 and y = 1. On the interval (−2, 1), the parabola has a greater x-value so the area between the two curves is $\int_{-2}^{1} \left[(3 - y^2) - (y + 1)\right]dy = \frac{9}{2}$

7. (C). When x < -1, |x + 1| = -x -1, and when x $\geq$ -1, |x + 1| = x + 1, so we have $\int_{-3}^{5} |x + 1|dx = \int_{-3}^{-1} (-x - 1)dx + \int_{-1}^{5} (x + 1)dx = 2 + 18 = 20$

8. (D). The velocity function v is equal $\int a(t)dt = t^2 + \frac{1}{4}\sin(4t) + C$ Use the fact that $v(0) = 0$ to get C = 0. The position function x is given by $\int v(t)dt = \frac{1}{3}t^3 - \frac{1}{16}\cos(4t) + C$ Here, use $x(0) = 0$ to find that $C = 1/16$. Thus, $x\left(\frac{\pi}{2}\right) = \frac{\pi^3}{24} - \frac{1}{16} + \frac{1}{16} = \frac{\pi^3}{24}$

9. (A). Notice that the integrand is the product rule applied to the function $P(x) = x^2 (\cos^3 x)$ Thus, the answer is $P(2\pi) - P(0) = 4\pi^2$.

10. (A). By the disc method, the desired volume is $\pi \int_{-1}^{0} \left(x\sqrt{x + 1}\right)^2 dx = \frac{\pi}{12}$

11. (B). Notice that the integrand is improper at $x = 3$, but this doesn't really matter because its antiderivative, $F(x) = 2\sqrt{x - 3}$, is defined there. The answer is F(7) - F(3) = 4 - 0 = 4.

12. (E). The function will be concave down when f00 < 0. By the Fundamental Theorem of Calculus, f'(x) = 2x(1 – 2x^2)^2. Taking derivatives again, we have f''(x) = 2(1 – 2x^2)^2 – 16x^2(1-2x^2). Setting this equal to 0, we get the solution set $\left\{\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{10}}\right\}$ Test around these points to see that f''(x) < 0 when

$$\frac{-1}{\sqrt{2}} < x < \frac{-1}{\sqrt{10}} \quad \text{or} \quad \frac{1}{\sqrt{10}} < x < \frac{1}{\sqrt{2}}$$

13. (A). Notice that the integrand is the quotient rule applied to $F(x) = \frac{\ln x}{e^x}$. The answer is $F(2) - F(1) = \frac{\ln 2}{e^2} - 0 = \frac{\ln 2}{e^2}$

14. (C). Integrate by parts with $u = x$ and $dv = f'(2x)$ to obtain

$$\int_2^5 xf'(2x)dx = \frac{x}{2}f(2x)\Big|_2^5 - \frac{1}{2}\int_2^5 f(2x)dx = \frac{5}{2}f(10) - f(4) - \frac{1}{2}\int_2^5 f(2x)dx =$$

$$5 - 0 - \frac{1}{2}\int_2^5 f(2x)dx = 5 - \frac{1}{2}\int_2^5 f(2x)dx$$ To obtain the integral on the right-hand side, set $u = 2x$ so that $du = 2dx$. Thus $\int_2^5 f(2x)dx = \frac{1}{2}\int_4^{10} f(u)du = \frac{1}{2}\left(\int_4^5 f(u)du + \int_5^{10} f(u)du\right) =$

$$\frac{1}{2}\left(\int_2^5 f(u)du - \int_2^4 f(u)du + \int_5^{10} f(u)du\right) = \frac{1}{2}(5 - 7 + 4) = 1$$ So the answer is 5 - 1/2 = 9/2.

15. (D). The given expression represents a Riemann sum of $f(x) = \frac{1}{1+x}$ on $0 \le x \le 1$, so the answer is $\int_0^1 \frac{1}{1+x}dx = \ln 2$

16. (C). The total distance is given by $\int_0^{100} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$, or $\int_0^{100} \sqrt{\left(\frac{2t}{t^2+1}\right)^2 + \left(1 - \frac{2}{t^2+1}\right)^2}\,dt =$

$$\int_0^{100} \sqrt{\frac{4t^2}{(t^2+1)^2} + \frac{(t^2-1)^2}{(t^2+1)^2}}\,dt = \int_0^{100} \sqrt{\frac{(t^2+1)^2}{(t^2+1)^2}}\,dt = \int_0^{100} 1\,dt = 100$$

17. (D). Graphing the function y = | cos x| for x $\ge$ 0, we see that it consists of repeats of the portion of y = | cos x| on the interval 0 $\le$ x $\le$ π. The area of this portion is equal to $2\int_0^{\pi/2} \cos x\,dx = 2$ thus F(n) = 2n and $\sum_{i=0}^{n} F(i) = \sum_{i=0}^{n} 2i = 2\sum_{i=0}^{n} i = 2\left(\frac{n(n+1)}{2}\right) = n^2 + n$

18. changed to E. Let $u^2 = x$ so that $2u\,du = dx$, and the integral becomes

$$\int \frac{\sin\sqrt{x}}{x}dx = \int \frac{\sin u}{u^2}(2u\,du) = 2\int \frac{\sin u}{u} = 2S(u) + C = 2S(\sqrt{x}) + C$$

19. (E). The graph of y = $10x - 16 - x^2$ is a concave-down parabola with roots at $x = 2$ and $x = 8$. From the graph, it's easy to see that $a = 2$ and $b = 8$; otherwise we'd be subtracting off area from that "hump" created by the parabola and the x-axis or only getting part of the hump. Thus, $a^3 + b^2 = 8 + 64 = 72$. (solution not changed)

20. (B). Use integration by parts repeatedly, always letting $dv = e^x$. When we do so, we get $\int P(x)e^x dx = P(x)e^x - P'(x)e^x + P''(x)e^x - P^{(3)}(x)e^x + \cdots$ Since P is a polynomial of degree 2004, derivatives of order 2005 and onward will be identically zero, yielding the answer given in B.

21. (D). Let $u = \frac{x}{n^2}$ so that $n^2\, du = dx$. The integral then becomes

$$\lim_{n\to\infty} \int_0^n f\left(\frac{x}{n^2}\right) dx = \lim_{n\to\infty}\left(n^2 \int_0^{1/n} f(u)\,du \right) = \lim_{n\to\infty} n^2 \sin\frac{1}{n^2} = \lim_{n\to\infty} \frac{\sin\frac{1}{n^2}}{\frac{1}{n^2}}$$

By L'Hˆopital's Rule, the above limit equals 1.

22. (E). All the answer choices could be elementary functions. For example, in choice A, choose F(t) = t^{-1} sin t and G(t) = -F(t). In choice B, let F(t) = t^{-1} sin t and G(t) = 1/F(t). As a counterexample to choice C, let F(t) = e^{t^2} and G(t) = $\sqrt{\ln t}$ Finally, letting F(t) = G(t) = t^{-1} sin t eliminates D.

23. (A). The area of R is equal to $\int_1^4 \frac{1}{x^2}\,dx = \frac{3}{4}$ Since the area of the rectangle with vertices at (1, 1/16), (4, 1/16), (4, 0), and (1, 0) has area of 3/16, we know that b > 1/16. Given that, the area of the subregion in R above $y = b$ is given by $\int_b^1 \left(\frac{1}{\sqrt{y}} - 1\right) dy$. Setting this integral equal to (3/4)/2 = 3/8 and solving for b, we find that $b = \frac{11}{8} - \frac{\sqrt{6}}{2}$.

24. (D). First, let $w^2 = x$, so $2w\,dw = dx$, thus transforming the integral into $\int_1^2 2we^w dw$. Now use integration by parts with $u = 2w$ and $dv = e^w$ to obtain

$$\int 2we^w dw = 2we^w \Big|_1^2 - \int_1^2 2e^w dw = 2we^w - 2e^w \Big|_1^2 = 2e^2$$

25. (B). By the standard formulas: $A = \int_0^1 \left(x^2 - x^4\right) dx = \frac{2}{15}$ $\quad m_y = \int_0^1 (x(x^2 - x^4))dx = \frac{1}{12}$

$$m_x = \frac{1}{2}\int ((x^2)^2 - (x^4)^2)dx = \frac{2}{45} \quad \text{Thus} \quad \bar{x} + \bar{y} = \frac{m_y}{A} + \frac{m_x}{A} = \frac{\frac{1}{12} + \frac{2}{45}}{\frac{2}{15}} = \frac{23}{24}$$

26. (A). Differentiating both sides of the equation, we get $2f(x)f'(x) = (f(x))^2 + (f'(x))^2$, or after collecting terms, $0 = (f(x) - f'(x))^2$. Thus, $f(x) = f'(x)$ The solution to this differential equation is none other than $f(x) = Ce^x$. Since f(0) = 1, this means C = 1, so f(x) = e^x. Thus, 100f(2) = $100e^2 \approx$ 738.9056099.

27. (C). Working out the details, we find that $f(x) = \begin{cases} 0 \text{ when } & 0 \le x < 1 \\ 1 \text{ when } & 1 \le x < 1 + \frac{1}{2} \\ 2 \text{ when } & 1 + \frac{1}{2} \le x < 1 + \frac{1}{2} + \frac{1}{4} \\ 3 \text{ when } & 1 + \frac{1}{2} + \frac{1}{4} \le x < 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \\ \vdots \end{cases}$

Consequently, the graph of *y = f(x)* looks like a staircase. The integral in question then represents the area created by the steps, or

$1(0) + \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{8}(3) + \cdots = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots$ Call this sum S. Then subtracting S/2 from S, we get $S = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots \quad \frac{S}{2} = \frac{1}{4} + \frac{2}{8} + \cdots \quad S - \frac{S}{2} = \frac{S}{2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{.5}{1 - .5} = 1$ making S = 2.

28. (C). Let A = (1, 1) and C = (5, 2). For some f $\in$ D, let x_0 be the zero of f on the interval (1, 5). Let B = (x_0, 0). Notice that F is precisely the arclength of f on the interval (1, 5). Since the shortest distance between two points is a straight line, we have $\int_1^5 \sqrt{1 + (f'(x))^2}\,dx = \int_1^{x_0} \sqrt{1 + (f'(x))^2}\,dx + \int_{x_0}^5 \sqrt{1 + (f'(x))^2}\,dx > AB + BC$

To minimize AB + BC, reflect A about the x-axis to obtain A_0. Note that the length of $A'C$ is the desired minimum distance, which is $\sqrt{(5-1)^2 + (2+1)^2} = 5$ (this procedure is similar to those going-home-but-must-stop-by-the-river-for-water types of problems).

29. (A). Note that $\int_0^n (7x^6 - 1)\,dx = n^7 - n$. By Fermat's Little Theorem, this expression is always divisible by 7 for any integer n. So the sum we want is the squares of all positive integers less than 50, or $1^2 + 2^2 + \cdots + 49^2 = 40425$. Note: Use $1 + 2^2 + 3^2 \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ to easily obtain the sum.

30. (E). The limit is pretty hard to evaluate by itself, so we'll estimate it instead. Note that since $f'(x) = \frac{1}{x^2 + (f(x))^2} > 0$, we must have *f* strictly increasing. Since f(0) = 1, f(x) > 1 if x > 0. Thus, $x^2 + (f(x))^2 > x^2 + 1$ if x is positive. So we have $\int_0^\infty f'(x)dx = \int_0^\infty \frac{1}{x^2 + (f(x))^2} < \int_0^\infty \frac{1}{x^2+1}dx = \arctan\infty - \arctan 0 = \frac{\pi}{2}$ and so $\int_0^\infty f'(x)dx = \left(\lim_{x\to\infty} f(x)\right) - f(0) < \frac{\pi}{2}$ Thus, $\lim_{x\to\infty} f(x) < \frac{\pi}{2} + f(0) = \frac{\pi}{2} + 1$. All of the answer choices are greater than this number. . .

TB 1 Ans: .79

$f(x+\Delta x) \approx f(x) + f'(x)\,dx$ $\quad \sin(.9) \approx \sin(\pi/4) + \cos(\pi/4)(.9-.7864) = .79$

TB2 Ans: $\frac{x^3}{3}\ln x - \frac{x^3}{9} + C$

Solution: $\int x^2 \ln x\,dx$ $\quad u = \ln x \quad du = \frac{1}{x}dx \quad x^2 = v' \quad v = \frac{x^3}{3}$

$$\frac{x^3}{3}\ln x - \int \frac{x^3}{3}\left(\frac{1}{x}\right)dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

TB3 Ans: $\frac{3}{\pi}$

Solution:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

$$f(c) = \frac{1}{\frac{\pi}{3} - \frac{\pi}{4}}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin 2x\,dx$$

$$f(c) = \frac{1}{\frac{\pi}{12}} \cdot \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \sin u\,du \quad u = 2x \quad du = 2dx \quad \text{when } x = \frac{\pi}{4} \quad u = \frac{\pi}{2} \quad \text{when } x = \frac{\pi}{3} \quad u = \frac{2\pi}{3}$$

$$f(c) = \frac{6}{\pi}\left[-\cos u\right]_{\frac{\pi}{2}}^{\frac{2\pi}{3}} = \frac{6}{\pi}\left[-\cos\frac{2\pi}{3} + \cos\frac{\pi}{2}\right] = \frac{6}{\pi}\left[\frac{1}{2} + 0\right] = \frac{3}{\pi}$$