1999 Integration Answer Sheet

1) The area of the region enclosed by the graphs of $y = e^x$, $y = 1$, and $x = 1$, is

$$\int_0^1 (e^x - 1)dx = e^x - x\Big]_0^1 = e - 1 - (1 - 0) = e - 2$$

Answer is (C)

2) $\int_0^2 e^{4x}dx =$

$$\int_0^2 e^{4x}dx = \frac{1}{4}e^{4x}\Big]_0^2 = \frac{1}{4}\left[e^8 - 1\right]$$

Answer is (A)

3) If $0 < k < \pi$, then $\int_0^k \cos(2x)dx = \frac{1}{2}$ when $k =$

$$\int_0^k Cos\ 2x\ dx = \frac{1}{2}Sin\ 2x\Big]_0^k = \frac{1}{2}[Sin\ 2k - Sin\ 0]$$

$$\frac{1}{2}Sin\ 2k = \frac{1}{2} \qquad \text{therefore} \qquad Sin\ 2k = 1$$

$$2k = \frac{\pi}{2} \quad or \quad k = \frac{\pi}{4}$$

Answer is (A)

4) The region in the first quadrant bounded by $y = \cos x$, $y = \sin x$, the y-axis, and the line $x = \frac{\pi}{4}$ is rotated about the x-axis. The volume of the resulting solid is

$$V = \int_0^{\frac{\pi}{4}} \left(\pi Cos^2 x - \pi Sin^2 x\right)dx =$$

$$\pi\int_0^{\frac{\pi}{4}} \left[\frac{1}{2}(1 + Cos\ 2x) - \frac{1}{2}(1 - Cos\ 2x)\right]dx =$$

$$\pi\int_0^{\frac{\pi}{4}} Cos\ 2x\ dx = \frac{\pi}{2}Sin\ 2x\Big]_0^{\frac{\pi}{4}} =$$

$$\frac{\pi}{2}\left[Sin\left(\frac{\pi}{2}\right) - Sin\ 0\right] = \frac{\pi}{2}$$

Answer is (A)

5) $\int_0^3 \frac{x}{x+1}dx$

$$\frac{x}{x+1} = 1 - \frac{1}{x+1}$$

$$\int_0^3 dx - \int_0^3 \frac{dx}{x+1} = x\Big]_0^3 - \ln|x+1|\Big]_0^3 =$$

$$3 - 0 - \ln 4 + \ln 1 = 3 - \ln 4 = 3 - 2\ln 2$$

Answer is (C)

6) Let $f(x)$ be the function defined by $f(x) = \begin{Bmatrix} x, & x \le 0 \\ x+1, & x > 0 \end{Bmatrix}$,

The value of $\int_{-2}^{1} x\, f(x)\,dx =$

$$\int_{-2}^{0} x(x)dx + \int_{0}^{1} x(x+1)dx =$$

$$\frac{x^3}{3}\Big]_{-2}^{0} + \frac{x^3}{3} + \frac{x^2}{2}\Big]_0^1 = 0 + \frac{8}{3} + \frac{1}{3} + \frac{1}{2} = \frac{7}{2}$$

Answer is (C)

7) $\int_e^{e^2} \frac{dx}{x(\ln x)^3} =$

$$\int_e^{e^2} (\ln x)^{-3}\left(\frac{1}{x}\right)dx = \int_1^2 y^{-3}dy$$

if $y = \ln x$, *when* $x = e$, $y = 1$ *and when* $x = e^2$, $y = 2$

Answer is (A)

8) $\int_{-3}^{3} \frac{dx}{x^2+9} =$

$$\frac{1}{3}\arctan\frac{x}{3}\Big]_{-3}^{3} = \frac{1}{3}[\arctan(1) - \arctan(-1)] =$$

$$\frac{1}{3}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi}{6}$$

Answer is (C)

9) $\int_2^4 |x-3|\,dx =$

$$\int_2^3 -(x-3)dx + \int_3^4 (x-3)dx =$$

$$\int_2^3 -\frac{x^2}{2} + 3x\Big]_2^3 + \quad \frac{x^2}{2} - 3x\Big]_3^4 =$$

$$-\frac{9}{2} + 9 + 2 - 6 + 8 - 12 - \frac{9}{2} + 9 = \frac{1}{2} + \frac{1}{2} = 1$$

Answer is (A)

10) The average value of the function $f(x) = \sin x$ on the closed interval $\left[0, \frac{\pi}{2}\right]$ is

$$AvgValue = \frac{1}{\frac{\pi}{2} - 0} \int_0^{\frac{\pi}{2}} \sin\, x\, dx =$$

$$\frac{2}{\pi}[-\cos x]_0^{\frac{\pi}{2}} = \frac{2}{\pi}[-0 + 1] = \frac{2}{\pi}$$

Answer is (D)

11) $\int_e^{e^2} \ln x\, dx =$

Using integration by parts

$$= x \ln x \quad - \int x\left(\frac{1}{x}\right)dx = x \ln x - x\Big]_e^{e^2} =$$

$$e^2 \ln e^2 - e^2 - (e \ln e - e) =$$

$$2e^2 - e^2 - e + e = e^2$$

Answer is (B)

12) If $\int_0^2 (2x^3 - kx^2 + 2k)dx = 12$, then k must be

$$\frac{2x^4}{4} - \frac{kx^3}{3} + 2kx = \frac{32}{4} - \frac{8k}{3} + 4k - 0 = 12$$

$$\frac{4}{3}k = 4 \qquad\qquad k = 3$$

Answer is (C)

13) Which definite integral represents the volume of a sphere with radius 2?

Volume by slicing gives

$$V = \int_{-2}^{2} \pi\left(\sqrt{4-x^2}\right)^2 dx = \pi\int_{-2}^{2}\left(4-x^2\right)dx = 2\pi\int_{0}^{2}\left(4-x^2\right)$$

Answer is (C)

14) Suppose $G(x) = \int_0^x \frac{1}{1+t^3}dt$ for all real x, then $G'(1) =$

$$G'(x) = \frac{1}{1+x^3} \qquad G'(1) = \frac{1}{2}$$

Answer is (B)

15) $\int xe^{3x}dx =$

Using integration by parts

$$= \frac{xe^{3x}}{3} - \int \frac{1}{3}e^{3x}dx = \frac{xe^{3x}}{3} - \frac{1}{9}e^{3x} + C = \frac{1}{3}e^{3x}\left[x - \frac{1}{3}\right] + C$$

Answer is (A)

16) For what value of k, $k > 0$, does $\int_0^k (4kx - 5k)\,dx = k^2$?

$$= \frac{4kx^2}{2} - 5kx\Big]_0^k = 2k^3 - 5k^2 - 0 = k^2$$

$$2k^3 - 6k^2 = 0$$

$$2k^2[k-3] = 0 \qquad k = 3$$

Answer is (C)

17) Suppose that the area under the curve $y = \frac{1}{x}$ from $x = a$ to $x = b$ is k.

The area, in terms of k, under the curve $y = \frac{1}{x}$ from $x = 2a$ to $x = 2b$ is

$$\int_a^b \frac{1}{x}dx = ln|x|\Big]_a^b = ln\left|\frac{b}{a}\right| = k$$

$$\int_{2a}^{2b} \frac{1}{x}dx = ln\,x\Big]_{2a}^{2b} = ln\left|\frac{2b}{2a}\right| = ln\left|\frac{b}{a}\right| = k$$

Answer is (A)

18) The region enclosed by the line $x + y = 1$ and the coordinate axes is rotated about the line $y = -1$. What is the volume of the solid genenerated?

By the washer method

$$\Delta V = \left(\pi R^2 - \pi r^2\right)dx = \pi\left[(2-x)^2 - 1^2\right]dx = \pi\left(x^2 - 4x + 3\right)dx$$

$$V = \pi\int_0^1 \left(x^2 - 4x + 3\right)dx = \pi\left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^1 = \frac{4}{3}\pi$$

Answer is (D)

19) Solve for x: $\int_{x^2}^{x^5} \frac{2}{t}dt = 54$

$$= 2\,ln|t|\Big]_{x^2}^{x^5} = 2\left[ln\left|x^5\right| - ln\left|x^2\right|\right] = 2\,ln\left|x^3\right| = 6\,ln|x|$$

$6\,ln|x| = 54 \qquad ln|x| = 9 \qquad x = e^9$

Answer is (A)

20) $\int_1^3 \frac{\sin\left(\frac{\pi}{x}\right)}{x^2}dx =$

$$= \frac{1}{\pi}Cos\left(\frac{\pi}{x}\right)\Bigg]_1^3 = \frac{1}{\pi}\left[Cos\left(\frac{\pi}{3}\right) - Cos\,\pi\right] = \frac{1}{\pi}\left[\frac{1}{2} - (-1)\right] = \frac{3}{2\pi}$$

Answer is (D)

21) If n is a positive integer, then $\int_0^{n\pi} |\sin x|\,dx$

$|\sin x$ *has a period of* $\pi|$ *and*

$$\int_0^{\pi} |\sin x|dx = -\cos x\Big]_0^{\pi} = -\cos\pi + \cos 0 = 2$$

$$\int_0^{n\pi} |\sin x|dx = n \textit{ times the area under the curve from } 0 \textit{ to } \pi = n\pi$$

Answer is (A)

22) $\int_2^4 \frac{x^2+2}{4x}dx =$

$$= \frac{1}{4}\int_2^4 x dx + \frac{1}{2}\int_2^4 \frac{dx}{x} = \frac{1}{8}x^2\Big]_2^4 + \frac{1}{2} ln|x|\Big]_2^4 = \frac{1}{8}[16-4] + \frac{1}{2}[ln 4 - ln 2] = \frac{3 + ln 2}{2}$$

Answer is (A)

23) The region bounded by the x - *axis* and the part of the graph of $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$ is divided into two regions by line $x = c$. If the area for the region for $0 \le x \le c$ is equal to the area of the region for $c \le x \le \frac{\pi}{2}$, then c must be

$$\int_0^c \cos x \, dx = \int_c^{\frac{\pi}{2}} \cos x \, dx$$

$$\sin x\Big]_0^c = \sin x\Big]_c^{\frac{\pi}{2}}$$

$$\sin c - \sin 0 = \sin \frac{\pi}{2} - \sin c$$

$$2 \sin c = 1 \qquad c = \frac{\pi}{6}$$

Answer is (B)

24) If the region bounded by the curve $f(x) = \sec x$, the x - *axis*, y - *axis*, and the line $x = \frac{\pi}{4}$, is revolved about the x - axis, what is the volume of the resulting solid?

$$\Delta V = \pi \sec^2 x \, dx$$

$$V = \pi \int_0^{\frac{\pi}{4}} \sec^2 x \, dx = \pi \tan x\Big]_0^{\frac{\pi}{4}} = \pi\left[\tan \frac{\pi}{4} - \tan 0\right] = \pi$$

Answer is (B)

25) If $f(x)$ is continuous on the interval $[a,b]$ and if $\int_a^x f(t)\,dt = 0$ for all x in $[a,b]$, then which of the following must be true?

I. f is constant on $[a,b]$

II. $f(x) \geq 0$ for all x in $[a,b]$

III. $f(x) = 0$ for all x in $[a,b]$

If antiderivative $= 0$ for all x in $[a,b]$ *then* $f(x) = 0$ *and* $I,\ II,$ *and* III *are true*

Answer is (D)

26) Which of the following is equal to $\int \frac{dx}{16 + x^2}$?

Answer is (C)

27) If $\int x \sec^2 x\,dx = f(x) + \ln |\cos x| + c$, then $f(x) =$

Using integration by parts

$\int x \sec^2 x dx = x \tan x - \int \tan x\,dx = x \tan x + \ln|\cos x| + C$

Answer is (C)

28) The Area of the region between the graph of

$y = 3x^2 + 3$ and the x - axis from x = 1 to x = 3 is

$\int_1^3 (3x^2 + 3)dx = x^3 + 3x]_1^3 = 27 + 9 - (1 + 3) = 32$

Answer is (B)

29) $\int_0^{\frac{\pi}{2}} \cos^2 x \ \sin x\,dx =$

$= -\frac{\cos^3 x}{3}\ ]_1^{\frac{\pi}{2}} = 0 + \frac{1}{3} = \frac{1}{3}$

Answer is (A)

30) $\int \frac{e^{\frac{-1}{x^2}}}{x^3}dx =$

$= \int e^{\frac{-1}{x^2}} \frac{dx}{x^3} = \frac{1}{2}e^{\frac{-1}{x^2}} + C$

Answer is (A)

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T. 1 $\frac{1}{3} \ln \left| \frac{x-1}{x+2} \right| + C$ OR $\frac{1}{3} \ln|x-1| - \frac{1}{3} \ln|x+2|$

T. 2 $1 - \frac{\pi}{2}$

T. 3 $\frac{1}{2}(1-e^{-6})$ or $\frac{1-e^{-6}}{2}$ OR $\frac{e^{6}-1}{2e^{6}}$