1992 National Mu Alpha Theta Convention

Integral Calculus Answer Key:

1.	A	16.	D
2.	C	17.	B
3.	A	18.	C
4.	C	19.	C
5.	D	20.	D
6.	A	21.	D
7.	B	22.	C
8.	D	23.	C
9.	A	24.	A
10.	B	25.	D
11.	B	26.	B
12.	D	27.	B
13.	D	28.	E
14.	C	29.	A
15.	C	30.	B

SOLUTIONS: INTEGRAL CALCULUS TOPIC TEST
1992 NATIONAL CONVENTION

A 1. $\int_0^1 x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1}$

C 2. $\int_0^{\pi/2} e^{\cos x}\sin x\,dx = -e^{\cos x}\Big|_0^{\pi/2} = -1+e = e-1$

A 3. Av. value $= \frac{1}{3}\int_1^4 \ln x\,dx = \frac{1}{3}\left[x\ln x - x\right]_1^4 = \frac{1}{3}\left[8\ln 2 - 4 + 1\right] = \frac{8\ln 2 - 3}{3}$

C 4. $\int_0^{2\pi}\cos^2 x\,dx$: $\cos^2 x$ has period of π, and average value of $\frac{1}{2}$.
So, since we are integrating over two full periods, Area $= \left(\frac{1}{2}\right)(2\pi - 0) = \pi$

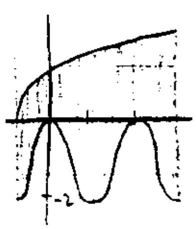
D 5. 

I won't work (negative area),
II will, III will because (area from 0 to 1) = (area from 3 to 4)

A 6. 

Area $= \int_0^1 (2x - x^2 - x^4)\,dx = x^2 - \frac{x^3}{3} - \frac{x^5}{5}\Big|_0^1 = \frac{7}{15}$

B 7. $\int_c^\infty \frac{dx}{c^2+x^2} = \frac{1}{c}\arctan\frac{x}{c}\Big|_c^\infty = \frac{\pi}{2c} - \frac{\pi}{4c} = \frac{\pi}{4c}$

D 8. 

Area same as $\int_{-1}^3 \sqrt{x+1}\,dx + (1)(4)$

$= \int_0^4 \sqrt{x}\,dx + 4$

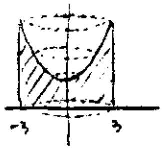
$= \frac{2}{3}x^{3/2}\Big|_0^4 + 4 = \frac{16}{3} + 4 = \frac{28}{3}$

A 9. Length $= \int_{\pi/6}^{\pi/3}\sqrt{1+\cot^2 x}\,dx = \int_{\pi/6}^{\pi/3}\csc x\,dx = \ln|\csc x - \cot x|\Big|_{\pi/6}^{\pi/3}$

$= \ln\frac{2/\sqrt{3} - 1/\sqrt{3}}{2 - \sqrt{3}} = \ln\frac{1}{2\sqrt{3}-3} = -\ln(2\sqrt{3}-3)$

B 10. $\int\frac{dx}{x-\sqrt{x}} = \int\frac{dx}{2\sqrt{x}}\,\frac{2}{\sqrt{x}-1} = 2\ln|\sqrt{x}-1| + c = \ln(\sqrt{x}-1)^2 + c$

B 11. $\int_{1/2}^{1} \frac{5^{1/x}}{x^2}\,dx$ let $u = \frac{1}{x}$. Then $\int_1^2 5^u\,du = \frac{5^u}{\ln 5}\Big|_1^2 = \frac{25-5}{\ln 5} = \frac{20}{\ln 5}$

D 12. 

$V = 2\pi\int_0^3 (x)(x^2+1)\,dx$ by shells

$= 2\pi\left(x^4/4 + x^2/2\right)\Big|_0^3 = \frac{99\pi}{2}$

D 13. The answer is D, all three: c can be anywhere in $(-\infty, \infty)$

C 14. $\int_0^1 \left(x^2 + \frac{3x+2}{x^2-x-12}\right)dx = \int_0^1 \left(x^2 + \frac{2}{x-4} + \frac{1}{x+3}\right)dx = \frac{x^3}{3} + \ln (x-4)^2(x+3)\Big|_0^1$

$= \frac{1}{3} + \ln 36 - \ln 48 = \frac{1}{3} + \ln\frac{3}{4}$

C 15. Work $= \int_0^{3\pi} \sin^2 x\,dx = (3\pi)\left(\frac{1}{2}\right)$ by the argument in problem 4

D 16. $\int_{-1}^{1} \frac{dx}{\sqrt{2^2+(x+1)^2}} = \operatorname{arcsinh}\left(\frac{x+1}{2}\right)\Big|_{-1}^{1} = \ln\left|x+\sqrt{x^2+1}\right|\Big|_0^1 = \ln\left(1+\sqrt{2}\right)$

or $\int_{-1}^{1} \frac{dx}{\sqrt{2^2+(x+1)^2}}$ let $2\tan t = x+1$. Then $2\sec^2 t\,dt = dx$

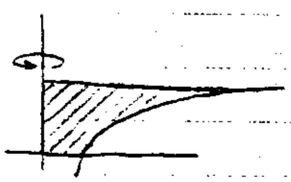
and we have $\int_0^{\pi/4} \frac{2\sec^2 t\,dt}{2\sec t} = \int_0^{\pi/4} \sec t\,dt$

$= \ln|\sec t + \tan t|\Big|_0^{\pi/4} = \ln\left|\sqrt{2}+1\right|$

B 17. $CM = \frac{\int_0^{10} x\,d(x)\,dx}{M} = \frac{\int_0^{10}(x^2+x)\,dx}{\int_0^{10} d(x)\,dx} = \frac{\frac{x^3}{3}+\frac{x^2}{2}\Big|_0^{10}}{\frac{x^2}{2}+x\Big|_0^{10}} = \frac{\frac{1150}{3}}{60} = \frac{115}{18}$

C 18. $\int x\tan^{-1}x\,dx = \frac{x^2}{2}\tan^{-1}x - \int \frac{x^2}{2}\frac{dx}{1+x^2} = \frac{1}{2}\left[x^2\tan^{-1}x - \int\left(1-\frac{1}{1+x^2}\right)dx\right]$

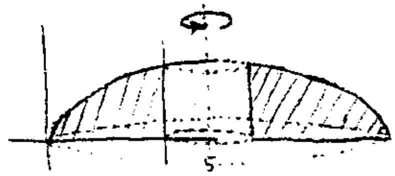
$= \frac{1}{2}\left[x^2\tan^{-1}x - x + \tan^{-1}x\right] = \frac{1}{2}\left[(x^2+1)\tan^{-1}x - x\right]$

C 19. 

Use y-axis integration: $\int_0^2 \pi\left(e^y\right)^2 dy$

$= \frac{\pi}{2}e^{2y}\Big|_0^2 = \frac{\pi}{2}\left[e^4 - 1\right]$

D 20. $\int \tan^3 x\,dx = \int \tan x \sec^2 x\,dx - \int \tan x\,dx = \frac{\tan^2 x}{2} + \ln|\cos x| + C$

D 21. 

$$V = 2\pi \int_0^4 (5-x)\sqrt{x}\,dx \quad \text{by shells}$$

$$= 2\pi\left[\frac{10}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^4 = 2\pi\left[\frac{80}{3} - \frac{64}{5}\right]$$

$$= 2\pi\left[\frac{400-192}{15}\right] = \frac{416\pi}{15}$$

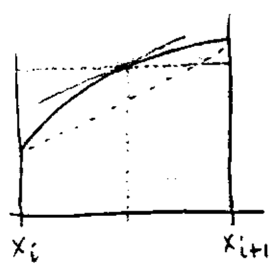
C 22. Top integral: $\int_0^{2\pi} \sin\left(\frac{x}{2}\right) dx = -2\cos\left(\frac{x}{2}\right)\Big|_0^{2\pi} = -2[-1-1] = 4$

Bottom integral: $\int_0^1 x^{-1/4}\,dx = \frac{4}{3}x^{3/4}\Big|_0^1 = 4/3$

Ratio $= 4/(4/3) = 3$

C 23. $\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx = \lim_{y\to1}\int_0^y \sqrt{\frac{1+x}{1-x}}\,dx = \lim_{y\to1}\int_0^y \frac{1+x}{\sqrt{1-x^2}}\,dx$

$$= \lim_{y\to1}\left[\sin^{-1}x - \sqrt{1-x^2}\right]_0^y = \left(\frac{\pi}{2}-0\right) - (0-1) = \frac{\pi+2}{2}$$

A 24. 

In every subinterval $T_i < I_i < M_i$ (ith subinterval)
so sum over all intervals yields

$$T < I < M.$$

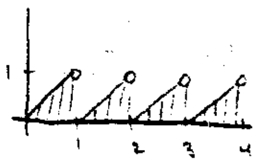
D 25. $a_y = te^t$ $\quad v_y = (t-1)e^t + C = (t-1)e^t + 1$

Then $p_y = (t-2)e^t + t + C = (t-2)e^t + t + 4$

Thus $P_y(2) = 6$ and since $6 = e^x + 1$, $x = \ln 5$. Thus

the particle is at $(\ln 5, 6)$

Note: that $y = e^x + 1$ is unimportant except to determine the particle's x-coordinate

B 26. 

So $\sum_{n=1}^{4} \int_0^n (x - [x])\,dx = \frac{1}{2} + 1 + \frac{3}{2} + 2 = 5$

B 27. The expression is essentially the definition of the integral

$$\int_0^3 (x^2 + x)\,dx = \frac{x^3}{3} + \frac{x^2}{2}\Big|_0^3 = 9 + \frac{9}{2} = \frac{27}{2}$$

E 28. Since the expression is odd $\left(\frac{\text{odd} \cdot \text{even}}{\text{even}} = \text{odd} \right)$

the integral equals zero

A 29. $\int_0^\infty x^n e^{-x}\,dx = -x^n e^{-x}\Big|_0^\infty + n\int_0^\infty x^{n-1} e^{-x}\,dx = n\int_0^\infty x^{n-1} e^{-x}\,dx$

Similarly $\int_0^\infty x^{n-1} e^{-x}\,dx = n-1\int_0^\infty x^{n-2} e^{-x}\,dx$

$$\text{So } \int_0^\infty x^n e^{-x}\,dx = (n)(n-1)(n-2)\ldots(1)\int_0^\infty e^{-x}\,dx$$
$$= (n!)\left(-e^{-x}\Big|_0^\infty\right)$$
$$= n!$$

[Note, this is the historic gamma function]

B 30. $A = \frac{1}{2}\int_0^{2\pi} r^2\,d\theta = \int_0^{2\pi} \left(2 + 2\cos\theta + \frac{1}{2}\cos^2\theta\right)d\theta$

$$= \int_0^{2\pi} 2\,d\theta + \int_0^{2\pi} 2\cos\theta\,d\theta + \frac{1}{2}\int_0^{2\pi} \cos^2\theta\,d\theta$$

$$= 4\pi + 0 + \frac{1}{2}\left(\frac{1}{2}\right)(2\pi) = 9\pi/2$$