MAΘ Nationals 2003 - Calculus Individual Solutions

① Profit = revenue − cost
$= x(100-x) - (x^2+4x+100)$
$= -2x^2 + 96x - 100$ which is maximized when $-4x+96=0$ or $x=24$; price is $100-24$
$= 76 \longrightarrow$ B/

② $\int\left(\frac{2}{x} + 1 - x^{-3/2}\right)dx$
$= 2\ln x + x + 2x^{-1/2} + C \rightarrow$ A/

③ $8 - x - x^2 = \frac{1}{2}x + \frac{7}{2} \Rightarrow x = -3$ or $3/2$
$dy/dx = -1-2x$; at $x=-3$, $dy/dx = -1-2(-3) = 5 \longrightarrow$ D/

④ $A = \pi r^2$; $dA/dr = 2\pi r$; $dA = 2\pi r\,dr$;
$dA = 2\pi(2)(.02) = 2\pi/25 \longrightarrow$ A/

⑤ $T = \frac{b-a}{2n}\left(y_0 + 2y_1 + 2y_2 + y_3\right)$
$= \frac{\pi/2}{2\cdot 3}\left(1 + 2\cdot\frac{\sqrt{3}}{2} + 2\cdot\frac{1}{2} + 0\right)$
$= \pi\left(2+\sqrt{3}\right)/12 \longrightarrow$ A/

⑥ I is false since $g(x)=0$ at $x=0$. II is false since $g(x)$ is decreasing on $(-2,0)$. III is false since $g''(x) = 3e^x(x^2+4x+2)$ changes sign twice. IV is false since $g(x)$ has a rel max at $x=-2$. $\longrightarrow$ E/

⑦ $\frac{du}{dx} = \frac{x}{\sqrt{x^2+9}}$, $\frac{dv}{dx} = 9x^2-2$;
$du/dv = \frac{du/dx}{dv/dx} = \frac{x}{(9x^2-2)\sqrt{x^2+9}}$
$= \frac{4}{(142)(5)} = \frac{2}{355} \longrightarrow$ D/

⑧ $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
$= \lim_{h\to 0}(6x+3h) = 6x$ so $f'(2) = 12$.
The antiderivative of $6x$ is $3x^2+c$, so $f(x) = 3x^2+c$ and since $f(1)=5$; $f(x) = 3x^2+2$ and $f(2)=14$.
$f(2)+f'(2) = 14+12 = 26. \longrightarrow$ D/

⑨ Examine the first few derivatives of $f(x)$.
$f'(x) = \frac{2}{2x+3}$;

⑨ (cont.) $f''(x) = \frac{-4}{(2x+3)^2}$;
$f'''(x) = \frac{16}{(2x+3)^3} \longrightarrow$ C/

⑩ Applying l'Hopital's Rule:
$\lim_{x\to 0}\frac{\sin 3x}{2x\,g(x)} = \lim_{x\to 0}\frac{3\cos 3x}{2x\,g'(x) + 2g(x)}$
$= \frac{3(1)}{2(0)(4)+2(-1)} = -3/2 \longrightarrow$ A/

⑪ $\frac{dV}{dt} = k(20{,}000-V)$; $\int(20{,}000-V)^{-1}dV = k\int dt$;
$-\ln|20{,}000-V| = kt+c$; $V = 20{,}000 - Ce^{-kt}$.
Since $V(0) = 2000$, $V = 20{,}000 - 18{,}000e^{-kt}$.
Since $V(1) = 3000$, $V = 20{,}000 - 18{,}000e^{t\ln 17/18}$.
Therefore $V(3) = 20{,}000 - 18{,}000e^{3\ln 17/18}$
$= 20{,}000 - 18{,}000\left(\frac{17}{18}\right)^3 \longrightarrow$ D/

⑫ $g'(x) = 3\sin^2(3x)$;
$g''(x) = 6\sin(3x)\cos(3x)\cdot 3 = 9\sin(6x)$;
$g'''(x) = 54\cos(6x) \longrightarrow$ A/

⑬ $A = \frac{1}{2}ab\sin C = \frac{1}{2}s^2\sin\theta$.
$\frac{dA}{dt} = \frac{1}{2}s^2\cos\theta\frac{d\theta}{dt} + s\frac{ds}{dt}\sin\theta$.
$= \frac{1}{2}(2\sqrt{3})^2(\sqrt{3}/2)(\pi/90) + (2\sqrt{3})(-\frac{1}{10})(\frac{1}{2})$
$= \frac{12\sqrt{3}\pi}{360} - \frac{\sqrt{3}}{10} = \frac{\sqrt{3}}{30}(\pi-3) \longrightarrow$ D/

⑭ Differentiating with respect to x:
$\cos x = 1 + x\sec^2 y\, dy/dx + \tan y$
so $\frac{dy}{dx} = \frac{\cos x - 1 - \tan y}{x\sec^2 y} = \frac{\cos x - \left(\frac{\sin x - x}{x}\right)}{x\sec^2 y}$
$= \frac{x\cos x - \sin x + x}{x^2\sec^2 y} \longrightarrow$ C/

⑮ $v(t) = 3(t-8)^{1/3} + C_1$; $v(9) = 6 \Rightarrow C_1 = 3$. $s(t) = \frac{9}{4}(t-8)^{4/3} + 3t + C_2$.
Since $v(t)$ changes sign when $t=7$,
total distance $= \left|\int_0^7 v(t)dt\right| + \left|\int_7^8 v(t)dt\right|$
$= \left|\left(\frac{93}{4}+C_2\right) - (36+C_2)\right| + \left|(24+C_2) - \left(\frac{93}{4}+C_2\right)\right|$
$= \frac{51}{4} + \frac{3}{4} = \frac{54}{4} = \frac{27}{2} \longrightarrow$ B/

⑯ Neither III nor V have derivatives at $x=0$. $x|x^2| = x^3$ whose 2^{nd} derivative is $6x$. Choices I & IV are both $|x^3|$ whose second derivative is $6|x|$. $\rightarrow$ B/

(17) $f_{avg} = \frac{1}{3-0}\left(\int_0^1 x^3 dx + \int_1^3 x^2 dx\right)$
$= \frac{1}{3}\left(\frac{1}{4}x^4\Big]_0^1 + \frac{1}{3}x^3\Big]_1^3\right) = \frac{107}{36}.$
Since $f(c) = \frac{107}{36}$, c must lie in the interval $[1,3]$ because $x^3 \le 1$ if $x \le 1$. Solving $c^2 = \frac{107}{36} \Rightarrow c = \frac{\sqrt{107}}{6}$ only. $\rightarrow$ C/

(18) $g(x) = h(1+f(x)) = (1+f(x))^2.$
$g'(x) = 2(1+f(x)) \cdot f'(x)$
$g'(1) = 2(1+f(1)) \cdot f'(1)$
$1 = 2(1+f(1)) \Rightarrow f(1) = -\frac{1}{2} \rightarrow$ B/

(19) $V_1 = \pi\int_0^4 x^3 dx = \frac{\pi}{4}x^4\Big]_0^4 = 64\pi.$
$V_2 = \pi\int_0^4 (64-x^3)dx = \pi\left(64x - \frac{1}{4}x^4\right)\Big]_0^4$
$= 192\pi.$ $V_1 : V_2 = 64\pi : 192\pi = 1:3 \rightarrow$ B/

(20) $f(x)$ is not continuous at $x=0$.
$g(x)$ is not differentiable at $x=2$.
$h(-2) \ne h(0)$ and $K(1) \ne K(3) \rightarrow$ E/

(21) $5\int (x^2+7)^{-2/3} x\, dx = \frac{5/2\,(x^2+7)^{1/3}}{1/3} + C$
$= \frac{15}{2}(x^2+7)^{1/3} + C \rightarrow$ C/

(22) $C_{avg} = \frac{1}{2-0}\int_0^2 \frac{1}{2}t\cos(t^2)dt$
$= \frac{1}{8}\sin(t^2)\Big]_0^2 = \frac{1}{8}\sin(4) \rightarrow$ E/

(23) $f(x)$ must satisfy 2 conditions. First, $f(0)=1$, which all choices meet. Secondly, $f'(x) = (f(x))^2+1$, which only IV meets since $\sec^2(x+\frac{\pi}{4}) = \tan^2(x+\frac{\pi}{4})+1 \rightarrow$ D/

(24) $y' = \frac{x^2-4x-4}{(x^2+4)^2}$; $y'' = \frac{-2x^3+12x^2+24x-16}{(x^2+4)^3}$
$y'' = \frac{-2(x+2)(x^2-8x+4)}{(x^2+4)^3} = 0$ at $x=-2$ or $x = 4 \pm 2\sqrt{3}$
Inflection pts are $(-2, \frac{1}{2})$, $(4+2\sqrt{3}, \frac{1-\sqrt{3}}{8})$ and $(4-2\sqrt{3}, \frac{1+\sqrt{3}}{8})$. The slope of the line joining these 3 collinear pts is $-\frac{1}{16}$. The equation of the line is $y - \frac{1}{2} = -\frac{1}{16}(x+2) \longrightarrow$ B/

(25) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{n^2}{n^2+k^2}$
$= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+k^2/n^2} = \int_0^1 \frac{dx}{1+x^2}$
$= \arctan x\Big]_0^1 = \arctan 1 = \pi/4 \longrightarrow$ C/

(26) $f(x) = \int -2x e^{-x^2} dx = e^{-x^2} + C$, through $(0,1) \Rightarrow C = 0$, so $f(x) = e^{-x^2}$.

(26) (cont.) Regard $g(x)$ as $\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$
$= F'(x)$ where $F(x) = \int_1^x e^{-t^2}dt$. So, $g(x) = e^{-x^2}$. Using the quotient rule,
$h(x) = \frac{d}{dx}\left(\frac{-e^{-x^2}}{2x}\right) = e^{-x^2}\left(1+\frac{1}{2x^2}\right).$ A/

(27) $Area = \int_b^1 3x^2 dx + \int_1^{b+2}(4-x)dx$
$= x^3\Big]_b^1 + \left(4x - \frac{1}{2}x^2\right)\Big]_1^{b+2} = -b^3 - \frac{1}{2}b^2 + 2b - \frac{9}{2}.$
$A' = -3b^2 - b + 2 = (2-3b)(b+1) = 0$ at $b = -1$ or $2/3$. Choose $b = 2/3$. B/

(28) The rate at which the disease is spreading is $f'(t) = 3Pe^{-t}(1+3e^{-t})^{-2}$.
This rate will be maximized when $\frac{d}{dt}f'(t) = f''(t) = 0$. $f''(t) = 3Pe^{-t}\ldots\ldots$
$(-2)(1+3e^{-t})^{-3}(-3e^{-t}) + (1+3e^{-t})^{-2}(-3Pe^{-t})$
$= 3Pe^{-t}(3e^{-t}-1)(1+3e^{-t})^{-3} = 0$ at $t = \ln 3$.
$\frac{f(\ln 3)}{P} = \frac{P}{(1+3(\frac{1}{3}))P} = 1/2 \longrightarrow$ C/

(29) $y' = -10(5x+2)^{-3}$; $m_{norm} = \frac{1}{10}(5x+2)^3 = \frac{25}{2} \Rightarrow x = \frac{3}{5}$, $y = \frac{1}{25}$;
$y - \frac{1}{25} = \frac{25}{2}\left(x - \frac{3}{5}\right)$ or $625x - 50y = 373$
$\longrightarrow$ A/

(30) Using u-substitution: let $u = \sqrt{2x-1}$
$\Rightarrow x = \frac{1}{2}(u^2+1) \Rightarrow dx = u\,du$
$\int_{1/2}^1 3x\sqrt{2x-1}\,dx = \frac{3}{2}\int_0^1 (u^4+u^2)du$
$= \left(\frac{3}{10}u^5 + \frac{1}{2}u^3\right)\Big]_0^1 = \left(\frac{3}{10}+\frac{1}{2}\right) - 0$
$= \frac{8}{10} = \frac{4}{5} \longrightarrow$ D/