Mu Alpha Theta National Convention 2004

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For all questions, answer E, "NOTA" means none of the above answers are correct

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Answers

#	Answer	#	Answer
1	D	18	Thrown out
2	D	19	C
3	C	20	D
4	A	21	A
5	A	22	E
6	D	23	D
7	E	24	C
8	D	25	C
9	A	26	B
10	B	27	C
11	C	28	B
12	B	29	B
13	C	30	A
14	D	TB1	3
15	D	TB2	$x \geq e$
16	D	TB3	
17	B		

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1. Ans: D $g'(x)=\dfrac{f'(x)}{f(x)}$ $g''(x)=\dfrac{f(x)f''(x)-(f'(x))^2}{(f(x))^2}$ $h(x)=g''(x)(f(x))^2$

and $h(x)=f(x)f''(x)-(f'(x))^2$

2. Ans: D $e^{\int dx}=e^x$ $e^x\dfrac{dy}{dx}+ye^x=xe^x$ $\dfrac{d(ye^x)}{dx}=\int xe^x dx$ integrating by parts gives

$ye^x=xe^x-e^x+C$ $y=x-1+Ce^{-x}$

3. Ans: C $v(t)=6t^2-8t+2$ $a(t)=12t-8=0$ $t=\dfrac{2}{3}$ $v\left(\dfrac{2}{3}\right)=-\dfrac{2}{3}\approx -0.67$

4. Ans: A $\pi\int_1^4 x^2dy=\pi\int_1^4 ydy=\pi\dfrac{y^2}{2}\Big|_1^4=\dfrac{15\pi}{2}$

5. Ans: A

$$\text{arc length}=\int_{-1}^{1}\sqrt{\left(\frac{1}{2}e^t-\frac{1}{2}e^{-t}\right)^2+\left(\frac{1}{2}e^t+\frac{1}{2}e^{-t}\right)^2}\,dt=\frac{\sqrt{2}}{2}\int_{-1}^{1}\sqrt{e^{2t}+e^{-2t}}\,dt=\sqrt{2}\int_0^1\sqrt{e^{2t}+e^{-2t}}\,dt$$

6. Ans: D Using L'Hopital's Rule, $\lim\limits_{h\to 0}\dfrac{\frac{1}{2}(1+h)^{-\frac{1}{2}}-\frac{1}{2}(1-h)^{-\frac{1}{2}}}{2h}=\lim\limits_{h\to 0}\dfrac{-\frac{1}{4}(1+h)^{-\frac{3}{2}}-\frac{1}{4}(1-h)^{-\frac{3}{2}}}{2}=-\dfrac{1}{4}$

7. Ans: B $S=2\pi\int_0^2(x^3-x)\sqrt{1+(3x^2-1)^2}\,dx=2\pi\int_0^2(x^3-x)\sqrt{9x^4-6x^2+2}\,dx$

8. Ans: D $\dfrac{dy}{dx}=f'(\sqrt{x})\dfrac{1}{2}x^{-\frac{1}{2}}$ $\dfrac{d^2y}{dx^2}=\left(f'(\sqrt{x})\right)\left(-\dfrac{1}{4}x^{-\frac{3}{2}}\right)+\dfrac{1}{2}x^{-\frac{1}{2}}f''(\sqrt{x})\dfrac{1}{2}x^{-\frac{1}{2}}$

$\left.\dfrac{d^2y}{dx^2}\right|_{x=4}=f'(2)\left(\dfrac{-1}{32}\right)+\dfrac{1}{16}f''(2)$

9. Ans: A $\lim\limits_{n\to\infty}\dfrac{1}{n}\left[\left(\dfrac{1}{\frac{1}{n}}\right)^2+\left(\dfrac{1}{\frac{2}{n}}\right)^2+\ldots+\left(\dfrac{1}{\frac{n}{n}}\right)^2\right]$ $\Delta x=\dfrac{b-a}{n}=\dfrac{1}{n}$ $f(x)=\dfrac{1}{x^2}$ the integral is $\int_0^1\dfrac{1}{x^2}dx$

10. Ans: B The area of each semicircle cross-section is

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$A = \frac{1}{2}\left(\pi y^2\right)$ $V = \int_o^1 A\,dx = \int_0^1 \frac{\pi}{2} y^2\,dx = \int_0^1 \frac{\pi}{2}\left(1-x\right)dx = \frac{\pi}{2}\left[x - \frac{1}{2}x^2\right]_0^1 = \frac{\pi}{4}$

11. Ans: C $\int_{-\infty}^{0} \frac{e^x}{e^{2x}+1}\,dx + \int_0^{\infty} \frac{e^x}{e^{2x}+1}dx = \lim_{t\to-\infty}\int_t^0 \frac{e^x}{e^{2x}+1}\,dx + \lim_{t\to\infty}\int_o^t \frac{e^x}{e^{2x}+1}\,dx$

$= \lim_{t\to-\infty} \arctan e^x \Big|_t^0 + \lim_{t\to\infty} \arctan e^x \Big|_0^t = \frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{2}$

12. Ans: B $6 * \frac{1}{2}\int_0^{\frac{\pi}{6}} a^2\left(\cos\left(3\theta\right)\right)^2 d\theta = 3a^2\int_0^{\frac{\pi}{6}}\left(\frac{1}{2} + \frac{1}{2}\cos\left(6\theta\right)\right)d\theta = \frac{\pi}{4}a^2$

13. Ans: C P = 800x – 35x^2 –(2x^3 – 140x^2 + 2600x +10000) $\frac{dP}{dx}$ = -6x^2 + 210x – 1800 = 0

x^2 – 35x + 300 = 0 (x – 20)(x – 15) = 0 x = 20, x = 15 To find the maximum,

$\frac{d^2P}{dx^2} = -12x + 210$ $\left.\frac{d^2P}{dx}\right|_{x=15} = 30$ $\left.\frac{d^2P}{dx^2}\right|_{x=20} = -30$ *Maximum* = 20

14. Ans: D $\frac{b-a}{n} = \frac{4-3}{4} = \frac{1}{4}$

$$A \approx \frac{1}{2}\left(\frac{1}{4}\right)\left[\left(3^3 - 3^2\right) + 2\left(3.25^3 - 3.25^2\right) + 2\left(3.5^3 - 3.5^2\right) + 2\left(3.75^3 - 3.75^2\right) + \left(4^3 - 4^2\right)\right] = 31.516$$

15. Ans: D Average value = $\frac{1}{e-1}\int_1^e x\ln x\,dx$ Integrating by parts gives

$\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\Big|_1^e = \left(\frac{e^2}{2}\ln e - \frac{e^2}{4}\right) - \left(\frac{1}{2}\ln 1 - \frac{1}{4}\right) = \frac{e^2+1}{4}$

16. Ans: D $\frac{dT}{dt} = k\left(T - S\right)$, where T is the temperature at a particular time, S is the surrounding temperature, and t is time. Using separation of variables, $\frac{dT}{\left(T-S\right)} = kdt$, and integration gives $\ln\left|T-S\right| = kt + C$ and $\left|T-S\right| = Ce^{kt}$. Substituting the values given in the problem of when t = 0, T=35, and S = 10, and when t = 5, T = 25, and S = 10 gives us the equation $\left|T-10\right| = 25e^{\frac{t}{5}\ln\left(\frac{3}{5}\right)}$.

Substituting T = 20 results in $t = 5\left(\frac{\ln\left(\frac{2}{5}\right)}{\ln\left(\frac{3}{5}\right)}\right) = 8.97$

17. Ans: B Using implicit differentiation, we find the $x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$ and $\frac{dy}{dx} = -\frac{y}{\left(x+2y\right)}$.

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At the point (1, 1), $\frac{dy}{dx} = -\frac{1}{3}$. Finding the second derivative, we have

$\frac{d^2y}{dx^2} = \frac{(x+2y)\left(\frac{dy}{dx}\right) - (y)\left(1+2\left(\frac{dy}{dx}\right)\right)}{(x+2y)^2}$. At the point (1, 1), $\frac{d^2y}{dx^2} = \frac{4}{27}$.

18. Ans: E

$v_1 = 2\cos 2t \quad v_2 = 4\cos t \quad a_1 = -4\sin 2t \quad a_2 = -4\sin t \quad -4\sin 2t = -4\sin t$ or $2\sin t\cos t = \sin t$

$2\sin t\cos t - \sin t = 0;\ \sin t(2\cos t - 1) = 0;\ \sin t = 0$ at $t = 0, \pi, 2\pi, \ldots;\ \cos t = \frac{1}{2}$ at $t = \frac{\pi}{3}, \frac{7\pi}{3}, \frac{13\pi}{3}, \ldots$

and at $t = \frac{5\pi}{3}, \frac{11\pi}{3}, \ldots$ The first positive value is $\frac{\pi}{3}$

19. Ans: C

$x_n = x_{n-1} + h$ and $y_n = y_{n-1} + hy'_{n-1}$ Our initial point is $(1,4)$ $y'(4) = 4$

$x_1 = 1.25 \quad y_1 = 4 + 0.25(4) = 5 \qquad x_2 = 1.50 \quad y_2 = 5 + 0.25(7) = 6.75$

$x_3 = 1.75 \quad y_3 = 6.75 + 0.25(12.25) = 9.8125 \qquad x_4 = 2 \quad y_4 = 9.8125 + 0.25(21.4375) = 15.171875$

20. Ans: D $h'(x) = f'(g(x)) * g'(x) \qquad h'(2) = f'(g(2)) * g'(2) = -1 * 4 = -4$

21. Ans: A $\frac{dy}{dx} = \frac{y}{1+x} \qquad \frac{dy}{y} = \frac{dx}{1+x} \qquad \ln|y| = \ln|1+x| + C$

$y = C(x+1) \quad 3 = C(1) \quad C = 3 \quad y = 3x + 3 \qquad f'(1) = 3$

$\int_0^{\frac{\pi}{4}} \tan^4 x \sec^4 x\,dx = \int_0^{\frac{\pi}{4}} \tan^4 x (1 + \tan^2 x) \sec^2 x\,dx = \int_0^{\frac{\pi}{4}} \tan^4 x \sec^2 x\,dx + \int_0^{\frac{\pi}{4}} \tan^6 x \sec^2 x\,dx =$

22. Ans: E

$\frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x \Big]_0^{\frac{\pi}{4}} = \frac{1}{5} + \frac{1}{7} = \frac{12}{35}$

23. Ans: D $g'(4) = \frac{1}{f'(g(4))} \qquad f'(x) = 2e^{2x} + 2e^x$ and $g(4) = 0 \qquad \frac{1}{f'(g(4))} = \frac{1}{2e^0 + 2e^0} = \frac{1}{4}$

24. Ans: C $y_{avg} = \frac{1}{3-0}\left(\int_0^1 x^3 dx + \int_1^3 x^2 dx\right) = \frac{1}{3}\left(\frac{1}{4} - 0 + 9 - \frac{1}{3}\right) = \frac{107}{36}$

25. Ans: C Since the bacteria grow exponentially, $y = y_0 e^{kt}$, where $y_0 = 5000$.

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$15{,}000 = 5000e^{10k} \quad 3 = e^{10k} \quad k = \frac{1}{10}\ln 3 \qquad y = 5000e^{\frac{1}{10}(\ln 3)(t)}$ when y=50,000

$10 = e^{\frac{1}{10}(\ln 3)(t)}$ or $t = \frac{10\ln 10}{\ln 3} \qquad a+b+c = 10+10+3 = 23$

26. Ans: B

$$P = \frac{\text{Area of region}}{\text{Area of square}} = \frac{\int_0^{2\sqrt{2}} \left(8 - x^2\right)dx}{64} \quad \left[8x - \frac{x^3}{3}\right]_0^{2\sqrt{2}} = 16\sqrt{2} - \frac{16\sqrt{2}}{3} = \frac{32\sqrt{2}}{3} \quad P = \frac{\frac{32\sqrt{2}}{3}}{64} = \frac{\sqrt{2}}{6}$$

27. Ans: C $\quad \int_{\ln 2}^{\ln 4} e^{2x+1}dx = \frac{1}{2}e^{2x+1}\Big]_{\ln 2}^{\ln 4} = \frac{1}{2}\left(e^{\ln 16+1} - e^{\ln 4+1}\right) = \frac{1}{2}\left(16e - 4e\right) = 6e$

28. Ans: B The dimensions of the rectangle are x by 15-x

$V_{cylinder} = \pi r^2 h = \pi x^2\left(15 - x\right) = \pi\left(15x^2 - x^3\right)$

$\frac{dV}{dx} = \pi\left(30x - 3x^2\right) = 3\pi x\left(10 - x\right) \quad \frac{dV}{dx} = 0$ when x=10 $\quad A_{\text{rectangle}} = (10)(5) = 50$

29. Ans: B

$f'(x) = 1 + 2\cos x = 0$ at $x = \frac{2\pi}{3} \qquad f'(x)$ is positive on $\left[\frac{\pi}{6}, \frac{2\pi}{3}\right)$ and negative on $\left(\frac{2\pi}{3}, \pi\right]$

Maximum occurs at $f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sqrt{3}$ Minimum occurs at $f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + 1$ (Less than $f(\pi) = \pi$)

Difference is $\frac{\pi}{2} + \sqrt{3} - 1$

30. Ans: A

$$R_1 = \int_{-1}^{2}\left(y + 2 - y^2\right)dy = \left(\frac{y^2}{2} + 2y - \frac{y^3}{3}\right)\Big]_{-1}^{2} = \frac{10}{3} - \left(-\frac{7}{6}\right) = \frac{27}{6}$$

$$R_2 = \int_0^1\left(y - y^2\right)dy = \left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \qquad \frac{27}{6} = k\left(\frac{1}{6}\right) \qquad k = 27$$

TB 1. Answer 3

$$\sqrt{\sin^2 x} = |\sin x| = \begin{cases} \sin x \text{ where } \sin x \ge 0 \\ -\sin x \text{ where } \sin x \le 0 \end{cases} \quad \int_{-\frac{\pi}{2}}^{\pi}|\sin x|\,dx = \int_{-\frac{\pi}{2}}^{0}\left(-\sin x\right)dx + \int_0^{\pi}\sin x dx = \cos x\Big]_{-\frac{\pi}{2}}^{0} + \left(-\cos x\right)\Big]_0^{\pi} =$$

$$\cos 0 - \cos\left(-\frac{\pi}{2}\right) + \left(-\cos\pi\right) - \left(-\cos 0\right) = 1 - 0 + 1 + 1 = 3$$

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TB 2. Answer $x > e$ or $x \geq e$

Solution: $f'(x) = \dfrac{x\left(\dfrac{8}{x}\right) - 8\ln x}{x^2} = 8\left(\dfrac{1-\ln x}{x^2}\right) = 0$ at $x = e$

$f'(x)$ is negative when $x > e$ and $f(x)$ is decreasing