GEMINI TOPIC TEST SOLUTIONS --- CALCULUS LEVEL
2000 NATIONAL MU ALPHA THETA CONVENTION

C 1. Make a system with $f(x)$ and $f(1/x)$ as your two variables. For second equation replace x with 1/x and eliminate $f(1/x)$ term. Thus, $f(x)$ is $\frac{2}{x^2}-x^2$. Answer is –4/3

D 2. The area under the standard normal distribution $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}dx = 1$. This the integral equals $\sqrt{2\pi}$

A 3. $arc = 4\int_0^{\pi/4}\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt$ Work below: Arc=4(10)=40

$$arc = 4\int_0^{\pi/2}\sqrt{(-5\sin t+5\sin(5t))^2+(5\cos t-5\cos(5t))^2}\,dt$$

C 4. Euler's formula: Faces+Vertices=Edges+2 Edges=26
Volume is $\sqrt{(12)(36)(48)}=144$. gcd(144,26)=2

B 5. Area is $.5\|(C-A)\times(B-A)\|$=3.5355 Volume is 1/6 of determinant of 4x4 matrix with last column set to 1. Value of B is 5.333. Greatest integer of A+B is 8

D 6. Make a polynomial in b and solve for positive root. b is 8

A 7. Use harmonic mean formula to determine height. If a=18 and b=22, then height of intersecting tie lines is given by $\frac{ab}{a+b}$. Height is 99/10. Taking log expression to answer gives 20.

C 8. Let x=2a, y=3b, substitute into equations. You can then transform this into a cubic polynomial $t^3-11t^2+24t+36=0$.

B 9. The coefficient is given by $\frac{10!}{2!2!2!4!}$=18900 Sum=18

D 10. Change x, y to x=3cos(t) and y=3sin(t) such that $0\le t\le \pi/3$.

$$S = 2\pi\int_0^{\pi/3}(3\sin t)\sqrt{(-3\sin t)^2+(3\cos t)^2}\,dt.\ \ S=9\pi$$

A 11. Extended Law of Sines: $Diameter = \frac{a}{\sin a} = \frac{b}{\sin b} = \frac{c}{\sin c}$

Diameter=$\frac{20}{\sqrt{3}}$ Area=$100\pi/3$

C 12. Loop bounded by $\theta=\pm\pi/4$ Area=$\frac{1}{2}\int_{-\pi/4}^{\pi/4}(\sec\theta-2\cos\theta)^2 d\theta = 2-\pi/2$

B 13. $u=-x^2$, $du=-2xdx$, $\frac{-1}{2}\int_{-1}^{-4}5^u\,du = \frac{62}{625\ln 5}$. A+b+c=692

A 14. Formula comes from analytic geometry. Make building y-axis and erect fence on positive x-axis. Construct hypotenuses and use derivatives to find minimum distances. Formula is $L=(a^{2/3}+b^{2/3})^{1.5}$. Answer is 33.30

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D 15. Surface Area=perimeter of 2D surface (π) times distance center travels around axis of rotation (π). Answer=π^2.

A 16. Center lies on (0,y) due to symmetry. CM=Moment about x-axis/total mass

$$Mass = \int_{-2}^{2}\int_{0}^{4-x^2} y\,dy\,dx = 256/15 \quad \text{Moment}=\int_{-2}^{2}\int_{0}^{4-x^2} y(y)\,dy\,dx = 4096/105 \quad \text{CM}=16/7$$

CM=(0,16/7) (Hardest question on test!)

C 17. Reverse S to be S= $n\binom{n}{n}+(n-1)\binom{n}{n-1}+\ldots$ Let $\binom{n}{k}=\binom{n}{n-k}$. Change terms accordingly. Match terms up for two equations in S and add to get $2S = n\binom{n}{0}+n\binom{n}{1}+n\binom{n}{2}+\ldots+n\binom{n}{n}$. Sum of S is then $S = n2^{n-1}$. If n=50, then S=50(2^{49})

C 18. First, expression for $\frac{\ln(t+1)}{t}$ is $1-t/2+t^2/6$...Take the integral, you have an x^3 term of $x^3/18$. At x=2, term is 4/9.

B 19. Series can be combined into $\sum_{n=1}^{\infty}\frac{2n}{3^{n+1}}$. Thus S=2/9+4/27+6/81...

Clever part: Subtract S from S/3 to get 2S/3=2/9+2/27+2/81...=1/3 S=1/2
Value of problem=3

D20. Change units first! $1.3(12^3) = 2246.4$ density. Then, find work to get all fluid to top of tank = $2246.4\int_{6}^{0} h(\frac{4\pi}{9})(6-h)^2\,dh + 2246.4(2)$ and raise it up 2 additional feet. (trick is turning radius feet into height feet and remembering to multiply by h (distance traveled)!)
Answer is 343242

A 21. $S=\sum_{k=0}^{25}\binom{25}{k}^2 = \binom{2n}{n} = 1.26\times10^{14}$ Thus S has 15 digits.

D 22. The expression $\frac{e^{ix}-e^{-ix}}{2} = \sin x, \frac{e^{ix}+e^{-ix}}{2} = \cos x$. This expression equals $\tan^2 x + 1 = \sec^2 x$

C 23. Okay, first let x=2t. Then f(4t)-f(0)= $16t^2$. Let f(0)=C. Now, f(4t)= $16t^2$+C. Next, x=t/4, substitute and f(x)= t^2+C. Derivative at x=.5 is 1

A 24. You have a sphere of radius 4. Translate center to origin. Next, find volume of both caps and the central shaft. Volume of central shaft is $\pi(1)^2(2\sqrt{15})$. Volume of two caps, you can find taking $2\pi\int_{0}^{4-\sqrt{15}}(\sqrt{16-(x-2)^2})^2\,dx$. Volume of remaining sphere is 234.

B 25. Value of encoding matrix is $\begin{bmatrix}15 & 5\\17 & 23\end{bmatrix}$. New encoded matrix is $\begin{bmatrix}339 & 321\\149 & 431\end{bmatrix}$. Sum of digits is 1240.

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B 26. Problem exactly like a venn diagram. Solution is
$7!-\binom{7}{1}6!+\binom{7}{2}5!-\binom{7}{3}4!+\binom{7}{4}3!-\binom{7}{5}2!+\binom{7}{6}1!-\binom{7}{7}0!=1854$

D 27.A=-9 Do system with points. You have a 3x3 with a cubic (since you have three points!). You should get $x^3-5x^2+7x-12=0$
Area of B intersection is 9/2. Centroid is (-1/2,12/5). B= $66\pi/\sqrt{29}$.

D 28. Number of terms in expansion is given by (#terms+exponent-1) C (#terms-1)
#terms=15. $\sum_{k=0}^{n} F_k = F_{k+2}-1$. $\frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^{n+1}-(\frac{1-\sqrt{5}}{2})^{n+1}) = F_n$ (n=17)
Sum=2584-1=2583

C 29. Translate ellipse to center: $\frac{x^2}{9}+\frac{y^2}{4}=1$ $Arc = 4\int_0^3 \sqrt{4(1-\frac{x^2}{9})} \approx 18.85$

A 30. Clever formula: using analytic geometry, you can extend pythagoras's theorem into 3D areas. Thus the sum of the squares of the 3 smaller triangular areas equals the square of the area of the larger triangle. Thus resulting area is $\sqrt{261}$. Of course, you're welcome to try heron's...