# 2003 Mu Alpha Theta National Convention
# Calculus Applications Topic Test – Mu Division

**Notes**: All functions and numbers in this test take on real number values.

1. Find the derivative of $f(x) = (x+1)(x+2)(x+3)$ at $x = 1$.

   (A) 0 (B) 13 (C) 26 (D) 39 (E) NOTA

2. Evaluate: $\lim_{x \to 0} \frac{\sin(5x)}{3x}$

   (A) 2 (B) $\frac{5}{3}$ (C) $\frac{4}{3}$ (D) 1 (E) NOTA

3. Find the average value of $e^x$ as $x$ increases from 1 to 3.

   (A) $\frac{e^3 - e}{2}$ (B) $e$ (C) $2e$ (D) $e^3$ (E) NOTA

4. Let $f(x) = 3x^2$. Evaluate: $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$

   (A) 15 (B) 14 (C) 13 (D) 12 (E) NOTA

5. Which of the following statements are always true?

   (A) If $f(x)$ is continuous everywhere, it is also differentiable everywhere.
   (B) If $f(x)$ is differentiable at $x = a$, it is also continuous there.
   (C) There exists a differentiable function that's not continuous.
   (D) There does not exist a continuous function that's not differentiable.
   (E) NOTA

6. Let $g(x) = \sin x$. Find the $2003^{\text{rd}}$ derivative of $g$ with respect to $x$.

   (A) $\cos x$ (B) $-\sin x$ (C) $-\cos x$ (D) $\sin x$ (E) NOTA

7. If $x$ is a real number, find the maximum value of $\sin x + \cos x$.

   (A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) 2 (D) $\sqrt{5}$ (E) NOTA

8. Find $y'$ if $y = x^3 \cos(4x)$.

   (A) $3x^2 \cos(4x) - 4x^3 \sin(4x)$ (B) $12x^2 \sin(4x)$

   (C) $3x^2 \sin(4x) - 4x^3 \cos(4x)$ (D) $3x^2 \cos(4x) + 4x^3 \sin(4x)$ (E) NOTA

9. Find the derivative of $h(x) = \sin^6 x + \cos^6 x + 3 \sin x \cos x (\sin^2 x + \cos^2 x)$ at $x = \pi/4$.

   (A) $\frac{\sqrt{2}}{2}$ (B) $\sqrt{3}$ (C) 0 (D) 1 (E) NOTA

10. Evaluate $\dfrac{dy}{dx}$ at $(2003, 2003)$ if $x^3 + y^3 = axy$, where $a$ is a constant not equal to 6009.

(A) 1 (B) $-1$ (C) 2 (D) $-2$ (E) NOTA

11. If $x(t) = (t^2 + 3t + 1)e^{2\ln t^2}$ describes the position of an antiparticle moving along the $x$-axis, find the acceleration of the antiparticle when $t = 1$.

(A) 5 (B) 25 (C) 61 (D) 102 (E) NOTA

12. If two resistors with resistances $R_1$ and $R_2$ (in ohms) are connected in parallel, then the total resistance R, measured in ohms, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If $R_1$ and $R_2$ are increasing at rates of 0.4 ohm per second and 0.5 ohm per second, respectively, how fast is $R$ changing in ohms per second when $R_1 = 100$ ohms and $R_2 = 200$ ohms?

(A) $\dfrac{7}{30}$ (B) $\dfrac{11}{18}$ (C) $\dfrac{8}{45}$ (D) $\dfrac{4}{9}$ (E) NOTA

13. A ladder 15 feet long rests against a vertical wall perpendicular to the ground. If the bottom of the ladder slides away from the wall at a speed of 3 feet per second, how fast is the angle between the top of the ladder and the wall changing in feet per second at the moment the angle is $\pi/4$ radian?

(A) $\dfrac{\sqrt{2}}{5}$ (B) 2 (C) $\dfrac{\sqrt{3}}{5}$ (D) $\dfrac{2}{5}$ (E) NOTA

14. Let $A(c)$ equal the area of the region bounded by the graph of $f(x) = x^4 + 2x^2 + 12$ and the $x$-axis from $x = 0$ to $x = c$, where $c \leq 100$. What is the value of $A'(5)$?

(A) 680 (B) 685 (C) 687 (D) 690 (E) NOTA

15. A telescope is sold at a price of $\displaystyle\sum_{x=1}^{2003} \frac{1}{x(x+1)}$ dollars. How much does the telescope cost, in dollars?

(A) 1 (B) 2 (C) 3 (D) 4 (E) NOTA

16. Let $a_n$ be a sequence given by $a_1 = 10^{-2003}$ and $a_{n+1}(a_n + 1) = a_n$ for $n \geq 1$. Which of the following is an upper bound for $\displaystyle\sum_{n=1}^{\infty} a_n$?

(A) 1 (B) $\dfrac{\pi}{2}$ (C) $\sqrt{e}$ (D) $5\ln 10$ (E) NOTA

17. Evaluate: $\lim_{x\to\pi/2} \dfrac{e^{\cos x}-1}{x-\pi/2}$

(A) $0$ (B) $-1$ (C) $1$ (D) $+\infty$ (E) NOTA

18. Which of the following intervals does $f(x) = x^3 - 26x + 1$ have a root?

(A) $0 < x < 5$ (B) $6 < x < 11$

(C) $12 < x < 17$ (D) $18 < x < 23$ (E) NOTA

19. Find the area of largest rectangle that can be inscribed in the graph of $y = -4x^2 + 64$ (where $y \geq 0$) and the $x$-axis given that one side of the rectangle lies on the $x$-axis.

(A) $\dfrac{2^7}{\sqrt{3}}$ (B) $\dfrac{2^8}{3}$ (C) $\dfrac{2^9}{27}$ (D) $\dfrac{2^{10}\sqrt{3}}{9}$ (E) NOTA

20. What is the maximum area of triangle $ABC$, given that $A = (0,0)$, $B = (0,10)$, and $C$ is a point on the curve given by $y = x^3 + 1$, where $|x| \leq 20$?

(A) 60 (B) 80 (C) 100 (D) 120 (E) NOTA

21. Suppose that $y = f(x)$ is a function such that $f'(x) = y$, and $f(5) = 3$. Find, to the nearest hundredth, the value of $f(6)$.

(A) 3 (B) 8.15 (C) 9.74 (D) 0 (E) NOTA

22. Find the volume of the solid obtained by rotating the region between the graphs of $y = x$ and $y = x^3$ about the $y$-axis.

(A) $\dfrac{\pi}{15}$ (B) $\dfrac{2\pi}{15}$ (C) $\dfrac{4\pi}{15}$ (D) $\dfrac{8\pi}{15}$ (E) NOTA

23. Find the point where the curves $y = x^3 - 3x + 4$ and $y = 3(x^2 - x)$ are tangent to each other.

(A) $(2,6)$ (B) $(-1,9)$ (C) $(-1,6)$ (D) $(2,9)$ (E) NOTA

24. What is the volume of solid formed when the circle given by $(x-6)^2 + y^2 = 1$ is revolved around the $y$-axis?

(A) $10\pi^2$ (B) $12\pi^2$ (C) $14\pi^2$ (D) $36\pi^2$ (E) NOTA

25. What is the area of an ellipse with major axis 10 and minor axis 6?

(A) $15\pi$ (B) $30\pi$ (C) $60\pi$ (D) $120\pi$ (E) NOTA

26. Find the area in-between the graphs of $y = x^2$ and $y = 10x$.

(A) 125 (B) $\dfrac{500}{3}$ (C) 250 (D) 500 (E) NOTA

27. Solid $A$ is 10 meters tall. The area of the cross sections of $A$ parallel to the ground is given by $C(x) = 3x^2 + 3$, where $x$ is the distance from the cross section to the ground in meters. What is the volume of $A$, in cubic meters?

(A) 1000 (B) 1010 (C) 1020 (D) 1030 (E) NOTA

28. Evaluate: $\lim_{x\to\infty} \left(\frac{2^x - 1}{2}\right)^{1/x}$

(A) 1 (B) 2 (C) $e$ (D) $e^2$ (E) NOTA

29. How many polynomials $P(x)$ with integer coefficients exist such that $P'(x) > 0$ and $(P(x))^2 + 4 \leq 4P(x^2)$ for all $x$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) NOTA

30. In the $xy$-plane, the points $(0, 0)$, $(2, 0)$, and $(0, \sqrt{3})$ are labelled $A$, $B$, and $C$ in some order. Find the sum of the coordinates of the point $P$ on the same plane inside triangle $ABC$ such that the sum of the distances from $P$ to each of $A$, $B$, and $C$ is as small as possible.

(A) $\frac{5+3\sqrt{3}}{13}$ (B) $\frac{1}{2} + \sqrt{\frac{5}{3}}$ (C) $1 + \frac{\sqrt{3}}{2}$ (D) $\frac{\sqrt{7}-2}{6}$ (E) NOTA

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# Mu Calculus Applications Topic Test – Solutions

1. **(C)**. By the Product Rule, $f'(x) = (x+2)(x+3) + (x+1)(x+3) + (x+1)(x+2)$, hence, $f'(1) = 12 + 8 + 6 = 26$.

2. **(B)**. Using L'Hôpital's Rule, $\lim_{x\to 0} \frac{\sin(5x)}{3x} = \lim_{x\to 0} \frac{5\cos(5x)}{3} = \frac{5}{3}$

3. **(A)**. Average value $= \frac{1}{3-1} \int_1^3 e^x \, dx = \frac{e^3 - e}{2}$

4. **(D)**. The given limit is simply the derivative of $f$ evaluated at $x = 2$. Since $f'(x) = 6x$, the answer is $6(2) = 12$.

5. **(B)**. Choice A and D are false (take $f(x) = |x|$, for example), as well as C because differentiability implies continuity, but that would mean B is true.

6. **(C)**. Note that $2003 \equiv 2004 - 1 \equiv 0 - 1 \equiv 3 \pmod 4$. Since $g$'s derivatives repeat at cycles of 4, the 2003rd derivative is the same as the third derivative, or $-\cos x$.

7. **(A)**. Note that $\sin x + \cos x = \sqrt{2}\sin(x + \pi/4)$. Since $|\sin u| \leq 1$, the maximum is $\sqrt{2}(1) = \sqrt{2}$.

8. **(A)**. By the Product Rule, $y' = 3x^2\cos(4x) - 4x^3\sin(4x)$.

9. **(C)**. Taking the derivative and using a lotta trig identities, the derivative reduces to $3\cos(2x)(\sin(2x) + 1)$ which, when evaluated at $x = \pi/4$, gives 0.

10. **(B)**. Differentiating implicity, we get $3x^2 + 3y^2y' = ay + axy'$, and after some rearrangement, $y' = -(3x^2 - ay)/(3y^2 - ax)$. Since $x = y = 2003$, $y'$ is just $-1$.

11. **(D)**. Simplify first to get $x(t) = t^6 + 3t^5 + t^4$, differentiate twice to get $x''(t) = a(t) = 12t^2 + 60t^3 + 30t^4$, and $a(1) = 102$.

12. **(A)**. First, we find the value of $R$, which is $((100)(200))/(100 + 200) = 200/3$. Then, differentiating the formula with respect to $t$, we get $R'/R^2 = R_1'/R_1^2 + R_2'/R_2^2$. Thus, $R' = (200/3)^2(.4/(100)^2 + .5/(200)^2) = 7/30$.

13. **(A)**. If $\theta$ is the angle between the top of the ladder and wall and $x$ is the distance from the wall to the base of the ladder, then the two variables can be related by $\sin\theta = x/15$. Differentiating, we get $(15\cos\theta)\theta' = dx$. Plug in values to get $\theta' = \sqrt{2}/5$.

14. **(C)**. Note that $f$ is nonnegative on the given interval. Thus, $A(c) = \int_0^c x^4 + 2x^2 + 12$, or $A'(c) = c^4 + 2c^2 + 12$, making $A'(5) = 687$.

15. **(E)**. Writing out the first few terms to try to find a pattern, we find that the $k$th partial sum is $k/(k+1)$. The $k$ in this problem is 2003, so the answer is 2003/2004.

16. **(E)**. Working out the first few terms reveals that $a_n = 1/(10^{2003} + n - 1)$. Since the sum of a harmonic series diverges, the given sum has no upper bound.

17. **(B)**. By L'Hôpital's Rule, $\lim\limits_{x \to \pi/2} \dfrac{e^{\cos x} - 1}{x - \pi/2} = \lim\limits_{x \to \pi/2} \dfrac{-(\sin x)e^{\cos x}}{1} = -1$

18. **(A)**. Since $f(0) = 0$ and $f(5) = -4$, $f$ has a root in-between that interval by the Intermediate Value Theorem. It's easy to check that the other intervals don't contain a root in them.

19. **(D)**. The area is given by $A(x) = 2xy = 2x(64 - 4x^2) = 128x - 8x^3$. Setting $A'(x) = 0$ yields a critical value of $x = 4/\sqrt{3}$, which is a maximum by the First Derivative Test. The answer is $A(4/\sqrt{3}) = 1024\sqrt{3}/9$.

20. **(C)**. By the matrix formula for the area of a triangle given three vertices, we get that the area expression in terms of $x$ (in fact, it's actually independent of $f(x)$!) is given by $5|x|$, making the maximum $5(20) = 100$.

21. **(B)**. The problem stated in the language of differential equations becomes $f'(x) = f(x)$, or after separation of variables, $f(x) = Ce^x$ for some constant $C$. Plug in the initial condition and get that $C = 3/e^5$, making $f(x) = 3e^{x-5}$, so $f(6) = 3e \approx 8.15$.

22. **(D)**. Set the $y$-values equal to each other to obtain intersection points of $(0,0)$, $(-1,-1)$, and $(1,1)$. Using symmetry and the Shell Method, the volume is $2\left(2\pi \int_0^1 x(x - x^3)\, dx\right) = 8\pi/15$.

23. **(A)**. In order for the two curves to be tangent to each other, they need to intersect and have equal derivatives at that intersection point. Solving $x^3 - 3x + 4 = 3(x^2 - x)$ and $3x^2 - 3 = 3(2x - 1)$ yields a common value of $x = 2$, so the point is $(2, 6)$.

24. **(B)**. The distance from the axis of revolution to the center of the circle is 6 while its area is $\pi$. By the Theorem of Pappus, the volume is $2\pi(6)(\pi) = 12\pi^2$.

25. **(A)**. The ellipse has a larger radius of 5 and smaller diameter of 3. By the standard formula the area is $\pi(5)(3) = 15\pi$.

26. **(B)**. The graphs intersect at $(0,0)$ and $(10,100)$, as easily shown by setting the equations equal to each other. The area is then $\int_0^{10} 10x - x^2\, dx = 500/3$.

27. **(D)**. Integration is the operation needed to sum up the cross sections. The volume is $\int_0^{10} 3x^2 + 3\, dx = 1030$.

28. **(B)**. Let the limit equal $L$. Take natural logs of both sides and write the limit as $\lim_{x\to\infty}(\ln(2^x - 1)/x - (\ln x)/x) = \ln L$. Using L'Hôpital's Rule separately, the left-hand side is equal to $\ln 2 - 0 = \ln 2$. Thus, $L = e^{\ln 2} = 2$.

29. **(A)**. Let $x = 0$ in the inequality to get $(P(0))^2 + 4 \leq 4P(0)$, or $(P(0) - 2)^2 \leq 0$. The square of any quantity can't be negative so we can conclude that $P(0) = 2$. Similarly, letting $x = 1$ yields $P(1) = 2$. Since $P(0) = P(1)$, there is a number $c$ in between 0 and 1 such that $P'(c) = 0$ (Rolle's Theorem), contradicting the fact that $P'(x) > 0$ for all $x$. No such polynomials $P$ exist.

30. **(A)**. Let $A = (0, 0)$, $B = (2, 0)$, and $C = (0, \sqrt{3})$. The point $P$ that minimizes the total distance is called the *Fermat Point* and is obtained by erecting equilateral triangles on each side of $ABC$ and finding the intersection point of each of the lines passing through a vertex of $ABC$ and the farthest equilateral triangle vertex opposite to it. After a lengthy calculation, we find that $P = (5/13, 3\sqrt{3}/13)$, making the answer A.