Advanced Calculus Solutions – Mu Level
2000 Mu Alpha Theta National Convention

1. The equation may be rewritten in the form: A

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Where a, b, and c are constants with $a^2 = b^2 = 3$ and $c^2 = 2$
Since a = b and c < a, the shape is an oblate spheroid.

2. $\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta+\rho^2\cos^2\phi-8\rho\sin\phi\cos\theta=0$

$\rho^2-8\rho\sin\phi\cos\theta=0$

$\rho=8\sin\phi\cos\theta$ C

3. I. is the definition of a conservative vector field. E

II. is incorrect because the line integral is independent of C.

III. follows from I. and is part of the definition of a conservative vector field.

4. $\frac{\partial f(x,y)}{\partial y}=2x^3e^{2xy}-\frac{x}{y^2}$ C

$\frac{\partial^2 f(x,y)}{\partial x\partial y}=6x^2e^{2xy}+4x^3ye^{2xy}-\frac{1}{y^2}$

$\frac{\partial^2 f(x,y)}{\partial x\partial y}=2x^2e^{2xy}(3+2xy)-\frac{1}{y^2}$

5. $V=xyz$

$dV=(yz)dx+(xz)dy+(xy)dz$ B

$dx=dy=dz=.02$

$dV=(.02)((12)(15)+(8)(15)+(8)(12))=7.92$

6. $\frac{\partial u}{\partial r}=\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial x}{\partial r}\right)+\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial y}{\partial r}\right)$ D

$\frac{\partial u}{\partial r}=(2x)(3)+(-2y)(1)$

$\frac{\partial u}{\partial r}=6x-2y=18r-6s-2r-4s=16r-10s$

7. $\nabla T(x,y)=(2x+y)\mathbf{i}+x\mathbf{j}$

$\nabla T(-3,1)=(-5)\mathbf{i}-3\mathbf{j}$ D

$\tan\theta=(y/x)=(-3)/(-5)=\frac{3}{5}$

$\theta=180+\tan^{-1}\left(\frac{3}{5}\right)=210.964$

8. Let $F(x,y,z)=x^2+y^2-3z-2$

$\nabla F(x,y,z)=2x\mathbf{i}+2y\mathbf{j}-3\mathbf{k}$

$\nabla F(-2,-4,6)=-4\mathbf{i}+-8\mathbf{j}-3\mathbf{k}$ B

Therefore the equation of the tangent plane may be obtained by:

$-4(x+2)-8(y+4)-3(z-6)=0$

$4x+8y+3z+22=0$

9. The surface is an elliptic paraboloid with a vertical axis, therefore any relative extremum will also be an absolute extremum over the domain of the function.

Let $f(x,y)=6x-4y-x^2-2y^2$ C

Then:

$f_{xx}(x,y)=-2 \qquad f_{yy}(x,y)=-4 \qquad f_{xy}(x,y)=0$

$f_{xx}(3,-1)f_{yy}(3,-1)-f_{xy}^{\,2}(3,-1)=(-2)(-4)-0=8>0$

and

$f_{xx}(3,-1)=-2<0,$

We conclude that (3,-1,11) is both a relative and absolute maximum of the function.

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10. Let $M = f_x(x,y) = 6x - 5y$

$N = f_y(x,y) = -5x + 6y^2$

$\therefore \nabla f(x,y) = M\mathbf{i} + N\mathbf{j}$

Integrating M with respect to x gives:

$f(x,y) = 3x^2 - 5xy + g(y)$ (Eq. 1)

$f_y(x,y) = -5x + g'(y)$ (Eq. 2)

By setting N equal to (2),

$g'(y) = 6y^2$

D

$g(y) = 2y^3 + C$

Substituting into (1) gives:

$f(x,y) = 3x^2 - 5xy + 2y^3 + C$

11. Only answer B describes an area of integration and a density formula which matches the given information. Note that C is incorrect as the correct polar integral would be:

$k\int_0^{\pi}\int_0^{2} r^2\,dr\,d\theta$ B

12. The triple integral in spherical coordinates takes the form

B

$\iiint f(\rho,\theta,\phi)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

and in this situation $f = \rho^2 \sin^2\phi \cos^2\theta$

13. The output of a dot product is always a scalar, so answers A and B are eliminated. Also, since **F** lies in the xy plane, answer C is eliminated. D has no such problems, and note that if $\mathbf{F} = 4e^{xy}\mathbf{i} - x^2y^2\mathbf{j}$, D

then $\nabla \bullet \mathbf{F} = 4ye^{xy} - 2x^2y$

14. Let C

$M = z^2$

$N = e^{-\cos(xy)}$

$R = \cos x^2$

$\mathbf{F} = M\mathbf{i} + N\mathbf{j} + R\mathbf{k}$

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

$$\therefore \alpha = \left(\frac{\partial M}{\partial z} - \frac{\partial R}{\partial x}\right) = (2z + 2x\sin x^2) = 2(z + x\sin x^2)$$

15. In (a):

$M = y$ A

$N = 2x$

$N_x - M_y = 2 - 1 = 1$

So by Green's theorem,

$\int_C y\,dx + 2x\,dy = \iint_R N_x - M_y\,dA = \iint_R dA =$ area of R

16. f increases as x^2, y^2, and $\ln(2.2z^2+.1)$ increase. Or, more simply, f increases as x, y, and z increase. Therefore P is located at the point in R at which x, y, and z are their greatest values: $(\sqrt{.5},\sqrt{.5}\sqrt{.5})$. The magnitude of the vector from (0, 0, 0) to P is given by: $\sqrt{.5+.5+.5} \approx 1.225$ D

17. Rewrite the first surface equation as $x^2 + y^2 - z^2 = 0$. By taking the gradient you find that vectors perpendicular to the first surface are <2x, 2y, -2z>. At $P_0(3,4,5)$, the vector is <6,8,-10> Employing the same approach with the second surface gives perpendicular vectors of <2x, 8y, 8z>, which at P_0 gives <6, 32, 40>. A vector parallel to the curve of intersection will be perpendicular to both curves' normal vectors. Therefore take the vector cross-product <6,8,-10> x <6,32,40> which gives <640,-300,144>

D

18. $\nabla F\big|_{(5,0,10)} = \left< 2x - \frac{y}{x^2}, \frac{1}{x}, 3z^2 \right>\Big|_{(5,0,10)} = \left< 10, \frac{1}{5}, 300 \right>$

$\vec{u} = \frac{< 2,5,-4 >}{\sqrt{2^2 + 5^2 + (-4)^2}} = \frac{1}{\sqrt{45}} < 2,5,-4 >$

$D_{\vec{u}} F = \nabla F \bullet \vec{u} = \frac{-1179}{\sqrt{45}}$

C

19. Use cylindrical coordinates with the rod's least-dense base on the xy plane and central axis running up the positive z axis.

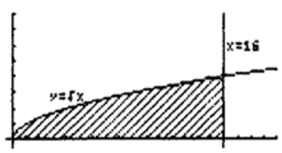
$I_0 = k \int_0^{10} \int_0^{2\pi} \int_0^{1} z^2 r^2 \, dr \, d\theta \, dz$

$I_0 = k \int_0^{10} \frac{2\pi z^2}{3} dz$

$I_0 = \frac{2000 k\pi}{9}$

D

20. The following graph shows the area of integration:



Which is correctly described by $\int_0^4 \int_{y^2}^{16} f(x,y)\,dx\,dy$

C

21. Since $\mathbf{F} = \nabla \phi$ where $\phi = \frac{2xy}{e} + 3y^2$

then $\int_C \mathbf{F} = \phi(x_2 y_2) - \phi(x_1 y_1)$

Let $\mathbf{R}(t) = M(t)\mathbf{i} + N(t)\mathbf{j}$

Then $x_2 = M(1) = 3e$, $x_1 = M(0) = 3$,

$y_2 = N(1) = \frac{4}{\pi}\tan^{-1} 1 = 1$, $y_1 = N(0) = 0$

$\phi(x_2 y_2) - \phi(x_1 y_1) = 6 + 3 - 0 - 0 = 9$

D

22. $F(x,y,z,\lambda) = x^2 + y^2 + z^2 + \lambda(3x - 2y + z - 4)$

$F_x = 2x + 3\lambda = 0, \quad F_y = 2y - 2\lambda, = 0, \quad F_z = 2z + \lambda = 0, \quad F_\lambda = 3x - 2y + z - 4 = 0$

$x = \frac{-3\lambda}{2}, \quad y = \lambda, \quad z = \frac{-\lambda}{2}$

$F_\lambda = \frac{-9\lambda}{2} - 2\lambda - \frac{\lambda}{2} - 4 = 0, \quad \therefore \lambda = \frac{-4}{7}, \quad x = \frac{6}{7}, \quad y = \frac{-4}{7}, \quad z = \frac{2}{7}$

A

23. B is undefined at (0,0), and therefore does not meet the requirements for a conservative vector field.

B

24. let $u = 3x, \ v = x^2$

$G(u,v) = \int_0^u e^{-vt^2} dt$

$\frac{\partial G}{\partial u} = e^{-vu^2}, \quad \frac{\partial G}{\partial v} = \int_0^u -t^2 e^{-vt^2} dt$

$\frac{du}{dx} = 3, \ \frac{dv}{dx} = 2x$

$\frac{dF}{dx} = \frac{dG}{dx} = \left(\frac{\partial G}{\partial u}\right)\left(\frac{du}{dx}\right) + \left(\frac{\partial G}{\partial v}\right)\left(\frac{dv}{dx}\right) = 3e^{-9x^4} + 2x \int_0^{3x} -t^2 e^{-x^2 t^2} dt$

A

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25. $A = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_R \sqrt{1 + x^2 + y^2}\, dA$

Use polar: $A = \int_0^{2\pi}\int_0^1 r\sqrt{1+r^2}\, dr d\theta$ (A)

$A = \int_0^{2\pi} \left.\frac{(1+r^2)^{\frac{3}{2}}}{3}\right|_0^1 d\theta = \int_0^{2\pi} \frac{2\sqrt{2}-1}{3} d\theta = \frac{2\pi}{3}\left(2\sqrt{2}-1\right)$

26. $f_x = 2x - y + 1 = 0$ $\quad f_{xx} = 2 \quad f_{xy} = -1$

$f_y = 2y - x - 5 = 0 \quad f_{yy} = 2$

$x = 1 \quad y = 3$ (B)

since $f_{xx}(1,3) > 0$ and $f_{xx}(1,3) \bullet f_{yy}(1,3) - f_{xy}^{\,2}(1,3) > 0$.

$f(x,y)$ has a minimum at $(1,3)$. $f(1,3) = -7$

27. $\frac{dV}{dt} = 16 - t^2$ (B)

$V = 16t - \frac{t^3}{3}$ (Eq.1)

$V = \pi\int_0^h y^2 dy = (1/3)\pi h^3$ (Eq. 2)

Setting h = 3.44, and (Eq. 2) = (Eq. 1)
we obtain 3.90

28. $A = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_1^{2\cos\theta} r dr d\theta$ (C)

$A = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{4\cos^2\theta - 1}{2} = 1.913$

29. Approaching (0,0) along the curve y=x, one obtains $\lim_{(x,y)\to(0,0)} f(x,y) = 1$ yet when approaching (0,0) along the curve x=0, one obtains $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ and since the limit is not the same along each curve, the limit does not exist. If the limit does not exist, then the function must be discontinuous at that point. (D)

30. $V = \frac{1}{2}\int_0^9 \left(\sqrt{y} - \frac{y}{3}\right)^2 dy$ (C)

$V = \frac{1}{2}\left.\left(\frac{y^2}{2} - \frac{4}{15}y^{\frac{5}{2}} + \frac{y^3}{27}\right)\right|_0^9 = \frac{27}{20}$