Advanced Calculus Examination

Each of the following problems has one correct answer out of five choices. The notation is standard. For example, $x = f^{-1}(y)$ is the inverse function of $y = f(x)$, where $f(x)$ is assumed to be restricted to values of x for which the inverse exists. The inverse of $y = \sin x$ is $x = \sin^{-1}(y)$ for $-\pi/2 \le x \le \pi/2$. Arcsin(y) is another name for $\sin^{-1}(y)$. Likewise $x = \tanh^{-1}(y)$ is the inverse function of $y = \tanh(x)$.
In a couple of problems you will have to use 62.5 lbs/ft^3 as the density of water.

1. Find the total force of the water on a dam if the water is 100 feet deep and the dam is in the shape of a trapezoid (i.e., the top and bottom edges are parallel) with bottom of length 200 feet and the water line 1100 feet long.

   a. 13750 tons
   b. 1375 tons
   c. 137.5 tons
   d. 7812.5 tons
   e. 78125 tons

2. Find the volume of the intersection of two circular cylinders, each of radius 1, if the center lines of the cylinders cross at a right angle.

   a. 16/3
   b. 8/3
   c. 8/3 π
   d. 4/3
   e. 2/3

3. Find the work done pumping water from one circular tank, of radius 10 feet and depth 5 feet, to an identical empty tank, the bottom of which is 20 feet above the bottom of the first tank. Assume that the pipe from one tank through the pump and to the other tank connects the bottoms of the tanks. Ignore the water in the pipe and pump. The units of the answers below are ft.-lbs.

   a. .625 e 5 π (i.e.: 62,500 π)
   b. .625 e 6 π
   c. .625 e 7 π
   d. .3125 e 8 π
   e. .3125 e 9 π

4. Find the volume of a donut whose surface is that generated by rotating a circle C of diameter 1 inch about a line 1.5 inches from the center of the circle C. The units of the answers below are $inch^3$.

   a. $3/4\ \pi$
   b. $3/4\ \pi^2$
   c. $3/4\ \pi^3$
   d. $4/3\ \pi$
   e. $4/3\ \pi^2$

5. A radioactive substance decays to 1/8 of its original mass in 24 years. What is its half life?

   a. 1 year
   b. 2 years
   c. 4 years
   d. 8 years
   e. 12 years

6. Newton's law of cooling states that the temperature T of a small hot object placed in a surrounding medium of constant temperature T_0 decreases at a rate proportional to $T - T_0$. If the object is initially at 290° C in air held at $T_0 = 20°$ C, and if it cools to 110° in one hour, what will its temperature be one hour later?

   a. 10° C
   b. 20° C
   c. 30° C
   d. 40° C
   e. 50° C

Problems 7-8: Evaluate the limits.

7. $\lim_{\theta \to \pi} [1 - \cos(\pi - \theta)]/(\pi - \theta)^2]$

   a. 1/2
   b. 1
   c. 2
   d. 4
   e. 8

8. $\lim_{x \to 0} (1 - \sin x)^{1/x}$

a. 1
b. -e
c. e
d. 1/e
e. -1/e

9. Calculate $\frac{d}{dy}(\tanh^{-1} y)$

a. $\dfrac{1}{1-y^2}$
b. $\dfrac{1}{1+y^2}$
c. $\dfrac{1}{\sqrt{1-y^2}}$
d. $\dfrac{1}{\sqrt{1+y^2}}$
e. $\ln|\cosh y|$

Problems 10-14: Solve the given integrals

10. $\int \tan^{-1} x \, dx$

a. $\tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$
b. $\tan^{-1} x + \frac{1}{2} \ln(1+x^2) + C$
c. $x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$
d. $x \tan^{-1} x + \frac{1}{2} \ln(1+x^2) + C$
e. $\ln|\sec x| + C$

11. $\int_{-\pi/2}^{\pi/2} |\sin x|^{1/2} \cos x \, dx$

a. 4/3
b. 2/3
c. 0
d. -2/3
e. -4/3

12. $\int \frac{x}{x^4 + 6x^2 + 10}\,dx$

a. $\tan^{-1}(x^2 + 3) + C$
b. $\frac{1}{2}\tan^{-1}(x^2 + 3) + C$
c. $\frac{1}{3}\tan^{-1}[(x^2 + 1)/3] + C$
d. $\frac{1}{6}\tan^{-1}[(x^2 + 1)/3] + C$
e. $\frac{1}{2}\tan^{-1}[(x^2 + 3)/2] + C$

13. $\int_1^{\infty} \frac{1}{x\sqrt{x^2 - 1}}\,dx$

a. $\pi/2$
b. $\pi/3$
c. $\pi/4$
d. $\pi/6$
e. ∞

14. $\int_0^4 \frac{1}{\sqrt{4x - x^2}}\,dx$

a. $-\infty$
b. ∞
c. 0
d. $\pi/2$
e. π

15. Evaluate the limit of the sequence $\{a_n\}$ where, for $n \geq 1$, $a_n = \frac{a_{n-1}^2 + 5}{2\,a_{n-1}}$, and a_n is always positive. You may (in fact you must) assume that $L = \lim_{n\to\infty} a_n$ exists in order to solve this problem. You don't need to know a_1.

a. $L = 0$
b. $L = 1$
c. $L = \sqrt{5}$
d. $L = \sqrt{5}/2$
e. $L = 5/2$

16. Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} (-1)^n \frac{(nx)^n}{n!}$$

a. $r = 1$
b. $r = 1/e$
c. $r = e$
d. $r = 0$
e. $r = \infty$

17. Use your calculator to evaluate $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{2n}}$, rounded to two decimal places.

a. .82
b. .88
c. .94
d. 1.00
e. 1.06

18. Find the Maclaurin series (i.e., the Taylor series about $x_o = 0$) for the function defined as

$$f(x) = \begin{cases} 1 & \text{for } x = 0 \\ \dfrac{\sin x}{x} & \text{for } x \neq 0 \end{cases}$$

a. $\sum_{n=0}^{\infty} (-1)^n \, x^{n+1}$
b. $\sum_{n=0}^{\infty} (-1)^n \, x^{2n}$
c. $\sum_{n=0}^{\infty} (-1)^n \, \frac{x^{2n+1}}{(2n)!}$
d. $\sum_{n=0}^{\infty} (-1)^n \, \frac{x^{2n}}{(2n)!}$
e. $\sum_{n=0}^{\infty} (-1)^n \, \frac{x^{2n}}{(2n+1)!}$

19. Find the distance of the point P(2, 3, 4) to the plane $4x + 3y + 12z = 0$.
a. 0
b. 1
c. 5
d. 13
e. 65

20. Find the volume of the parallelopiped determined by the vectors

$$\mathbf{A} = \mathbf{i} - \mathbf{j} - \mathbf{k}$$
$$\mathbf{B} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$
$$\mathbf{C} = \mathbf{i} + \mathbf{j} - \mathbf{k}$$

a. 0
b. 2
c. –2
d. 4
e. –4

21. A function f(x, y, z) satisfies Laplace's equation in dimension 3 if $f_{xx} + f_{yy} + f_{zz} = 0$. This equation arises in the study of many phenomena, such as, e.g., the steady state distribution of temperatures in a solid body. Find the values of n for which

$$f(x, y, z) = (x^2 + y^2 + z^2)^n$$

satisfies Laplace's equation away from the origin, that is, for $x^2 + y^2 + z^2 \neq 0$.

a. n = 0 or 1/2
b. n = 0 or -1/2
c. n = 0 only
d. n = -1/2 only
e. n = 1/2 only

22. Find parametric equations for the tangent at (x, y, z) = (1, 1, 1) to the curve defined as the curve satisfying both the equations $x^3 + y^2 + z^3 = 3$, and xyz = 1.

a. x = 1, y = -t + 1, z = t + 3
b. x = t + 1, y = t - 1, z = t + 1
c. x = -t + 1, y = 1, z = t + 1
d. x = 3t + 1, y = 2t + 1, z = 3t + 1
e. x = t + 1, y = t + 1, z = t + 1

23. Find the absolute maximum value of $f(x, y) = x^2 y^2$ on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

a. 0
b. 1
c. 10
d. 100
e. 1000

24. Find the coordinates of the critical points of the function $f(x, y) = x^3 + y^3 - 3xy + 6$ and identify each as a saddle point or as the location of a local minimum or a local maximum value of $f(x, y)$.

	saddle point	local minimum	local maximum
a.	(0, 0)	none	(1, 1)
b.	(0, 0)	(1, 1)	none
c.	(1, 1)	(0, 0)	none
d.	(1, 1)	none	(0, 0)
e.	none	(1, 1)	(0, 0)

25. Find the volume of the prism in space bounded by the planes $x = 0$, $y = 0$, $x + y = 1$, $z = 0$, and $x + y + z = 10$.

a. 14/3
b. 25/6
c. 7/2
d. 3
e. 5/2

26. Find the mass of a portion of the solid sphere centered at the origin and of radius 5, where the portion is the part of the sphere that lies between the cones $\phi = \pi/3$, and $\phi = 2\pi/3$, in spherical coordinates (ρ, ϕ, θ). The density of the sphere at a point with coordinates (ρ, ϕ, θ) is ρ^2.

a. $62.5\,\pi$
b. $625\,\pi$
c. $1250\,\pi$
d. $\frac{500}{3}\pi$
e. $\frac{12500}{3}\pi$

27. Let R be the parallelogram in the xy–plane bounded by the lines

$$y = -3x \quad \text{and} \quad y = 3x$$
$$y = -3x + 4 \quad \text{and} \quad y = 3x + 3$$

Evaluate the double integral $\iint_R 9x^2 - y^2 + 5\, dV$, by an appropriate change of variables.

a. 4
b. 5
c. 6
d. 7
e. 8

Tie Breakers.

The solutions to the following problems must include justification of your procedures.

28. Establish the convergence or divergence of the sequence $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$, for $n = 1, 2, \cdots$.

    My work for problem 28. can be found on the back of page _____ .

29. Determine the area enclosed by the curve given parametrically by the equations

$$x(t) = (\cos t)(1 + \sin t)$$
$$y(t) = \sin t$$

    Here $0 \le t \le 2\pi$.

    My work for problem 29. can be found on the back of page _____ .