1992 National Mu Alpha Theta Convention

Trigonometry Topic Test Answers:

1. D	16. C
2. B	17. C
3. B	18. B
4. A	19. D
5. D	20. A
6. D	21. D
7. C	22. A
8. C	23. C
9. C	24. D
10. C	25. D
11. A	26. D
12. C	27. C
13. A	28. B
14. A	29. A
15. D	30. A

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1992 MAΘ Convention
TRIGONOMETRY TEST
Solutions

D 1. $\tan\left(q+\frac{19\pi}{2}\right)=\tan\left(q+\frac{\pi}{2}\right)=\frac{\sin\left(q+\frac{\pi}{2}\right)}{\cos\left(q+\frac{\pi}{2}\right)}=\frac{\cos q}{-\sin q}=-\cot q=\boxed{\frac{-1}{b}}$

B 2. sine is positive, so cosecant is positive.
cosine is negative, so secant is negative.
tangent is negative, so cotangent is negative.
The answer is $\boxed{2}$.

B 3. $\tan 2a=\frac{2\tan a}{1-\tan^2 a}=\frac{8}{1-16}=\boxed{-\frac{8}{15}}$

A 4. $\sec 2a=\frac{1}{\cos 2a}=\frac{1}{1-2\sin^2 a}=\frac{1}{1-2\left(\frac{9}{16}\right)}=\frac{1}{1-\frac{18}{16}}=\frac{1}{-\frac{2}{16}}=\boxed{-8}$

D 5. $\cos(\pi-a)=\cos\pi\cos a+\sin\pi\sin a=\boxed{-\cos a.}$

D 6. Secant (x) ranges from ± 1 to $\pm\infty$, so that $17\sec(3x)$ ranges from $-\infty$ to -17 and 17 to ∞. The answer is $\boxed{(-\infty,-17] \text{ and } [17,\infty).}$

C 7. $\sec x=2\sin x\tan x=2\sin x\frac{\sin x}{\cos x}=\frac{2\sin^2 x}{\cos x}$

$\frac{1}{\cos x}=\frac{2\sin^2 x}{\cos x}\Rightarrow 1=2\sin^2 x$ (since $\cos x\neq 0$) $\Rightarrow \sin x=\frac{\sqrt{2}}{2}$

$\Rightarrow x=\boxed{\frac{\pi}{4}}$

C 8. $\sin(-x)=-\sin x$ $\quad\cos(-x)=\cos x$ $\quad\sec(-x)=\sec x$ $\quad\csc(-x)=-\csc x$
$\tan(-x)=-\tan x$ $\quad\cot(-x)=-\cot x$. Combining these we can see that i), iv) work and ii), iii) don't. $\boxed{\text{Two}}$ is the answer.

C 9. $\sqrt{\dfrac{2\sin x - \cos x \sin 2x}{\sin 2x \sec x}} = \sqrt{\dfrac{2\cos x \sin x - \cos^2 x \sin 2x}{\sin 2x}} = \sqrt{1-\cos^2 x} = \boxed{|\sin x|.}$

C 10. $\sin 4x\pi$ repeats every $\frac{1}{2}$, since $4\pi(\frac{1}{2}) = 2\pi$; $\cos 5x\pi$ repeats every $\frac{2}{5}$. (The amplitudes and phases are irrelevant to the period.) Thus the sine term is the same at $\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots,$ and the cosine term at $\frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \frac{8}{5}, 2, \ldots$; the first time they repeat at the same time is the period, which is at $\boxed{2.}$

A 11. The distance from (x_1, y_1) to the origin is $\sqrt{x_1^2+y_1^2}$

$$= \sqrt{(a^2 x_0^2 \cos^2\theta + 2a^2 x_0 y_0 \cos\theta \sin\theta + a^2 y_0^2 \sin^2\theta) + (a^2 x_0^2 \sin^2\theta - 2a^2 x_0 y_0 \cos\theta \sin\theta + a^2 y_0^2 \cos^2\theta)}$$

$= \sqrt{a^2 x_0^2 + a^2 y_0^2} = a\sqrt{x_0^2 + y_0^2} = \boxed{ar_0.}$ Note that if we notice that the transformation $(x_0, y_0) \to (x_1, y_1)$ is just a rotation centered at the origin plus a dilation centered at the origin the result falls out without calculation.

C 12. $\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \boxed{\dfrac{\sqrt{6}-\sqrt{2}}{4}.}$

A 13. The law of cosines gives

$$(BC)^2 = (AB)^2 + (AC)^2 - 2(AB)(AC)\cos\angle A,$$

or

$$81 = 16 + 49 - 56\cos\angle A$$

$$56\cos\angle A = -16 \Rightarrow \cos\angle A = \boxed{-\frac{2}{7}.}$$

A 14. The extension to the law of sines states that

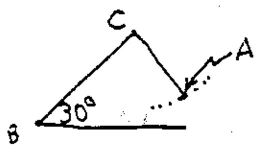
$\dfrac{x}{\sin X} = D$, where x is any side and X the corresponding angle. Thus $D = \dfrac{BC}{\sin\angle A} = \boxed{18.}$

P 15. We could directly extract $\sin q$ from $\sin 2q$ and evaluate, but this is simpler:

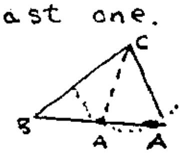
$$\sin^2 q+\cos^2 q+\sec^2 q+\tan^2 q+\csc^2 q+\cot^2 q$$
$$=1+\frac{1+\sin^2 q}{\cos^2 q}+\frac{1+\cos^2 q}{\sin^2 q}$$
$$=1+\frac{1+\sin^4 q+\cos^4 q}{\sin^2 q\cos^2 q}=1+\frac{1+(\sin^2 q+\cos^2 q)^2-2\sin^2 q\cos^2 q}{\sin^2 q\cos^2 q}$$

and since $\sin^2 q\cos^2 q=\frac{\sin^2 2q}{4}=\frac{3}{25}$, this is $1+\frac{25}{3}(2-\frac{6}{25})=\boxed{\frac{47}{3}}$.

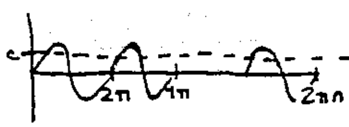
C 16.



The altitude is $BC\sin B=6$, which is less than AC, so there is at least one. Also $AC<BC$, so there are $\boxed{\text{two}}$.



C 17. We have $\sin n\theta=c$ for $0\le\theta<2\pi\Rightarrow 0\le n\theta<2\pi n$.
For each 2π, there are two values $\Rightarrow$ the answer is $\boxed{2n.}$



$\rightarrow 2n$ values

B 18.
$$x_{n+2}=\sin(\mathrm{Cos}^{-1}(\sin(\mathrm{Cos}^{-1}x_n)))$$
$$=\sin(\mathrm{Cos}^{-1}(\sin(\tfrac{\pi}{2}-\mathrm{Sin}^{-1}x_n)))$$
$$=\sin(\mathrm{Cos}^{-1}(\cos(\mathrm{Sin}^{-1}x_n)))=\sin(\mathrm{Sin}^{-1}x_n)=x_n.$$
Thus $x_{85}=x_{83}=\cdots=x_1=\sin(\mathrm{Cos}^{-1}\frac{5}{13})=\sin(\mathrm{Sin}^{-1}\frac{12}{13})=\boxed{\frac{12}{13}}$.

D 19.
$$x+y=\cos(M+N)+\sin(M-N)=\cos M\cos N-\sin M\sin N+\sin M\cos N-\sin N\cos M$$
$$=(\cos M+\sin M)(\cos N-\sin N)$$
$$=0,$$

so that $\cos M=-\sin M\Rightarrow M=\frac{3\pi}{4},\frac{7\pi}{4}$

or $\cos N=\sin N\Rightarrow N=\frac{\pi}{4},\frac{5\pi}{4}\Rightarrow$ in answer $\boxed{D}$ $N=\frac{\pi}{4}$.

A 20. We have $F(-\cos a) = F(\cos(\pi - a)) = \cos 7(\pi - a)$

$$= \cos(7\pi - 7a) = \cos(\pi - 7a) = -\cos 7a$$
$$= -F(\cos a).$$

Thus F is an odd polynomial, so, without calculation, we know the x^2 coefficient, like all even powers, is $\boxed{0.}$

D 21. $y = |\tan t| = \left|\frac{\sin t}{\cos t}\right| = \frac{|x|}{\sqrt{1-x^2}}$, $-1 < x < 1$. The domain is thus bounded, whereas the range goes from 0 to ∞, with asymptotes at $x = \pm 1$.

A 22. $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\cos\frac{4\pi}{7}\cos\frac{5\pi}{7}\cos\frac{6\pi}{7}$

$$= -\left(\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right)^2.$$

But $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \dfrac{\sin\frac{\pi}{7}\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}}{\sin\frac{\pi}{7}}$

$$= \frac{\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}}{2\sin\frac{\pi}{7}} = \frac{\sin\frac{4\pi}{7}\cos\frac{3\pi}{7}}{4\sin\frac{\pi}{7}}$$

$$= \frac{\sin\frac{3\pi}{7}\cos\frac{3\pi}{7}}{4\sin\frac{\pi}{7}} = \frac{\sin\frac{6\pi}{7}}{8\sin\frac{\pi}{7}} = \frac{1}{8},$$ so the answer is $\boxed{-\frac{1}{64}}$

C 23. $0 < \sin v < \cos w \Rightarrow$ v, w in Quadrant I.

$\cos w > \sin v = \cos(90° - v) \Rightarrow w < 90° - v \Rightarrow \boxed{v + w < 90°}$

D 24. $\cos 3x = \sin 2x \Rightarrow 4\cos^3 x - 3\cos x = 2\sin x\cos x$

$\Rightarrow \cos x = 0$ OR $4\cos^2 x - 3 = 2\sin x$

$$-4\sin^2 x + 1 = 2\sin x$$
$$4\sin^2 x + 2\sin x - 1 = 0$$
$$\sin x = \frac{-2 \pm 2\sqrt{5}}{8}$$

For $\cos x = 0$, $x = \frac{\pi}{2}, \frac{3\pi}{2}$. For each of the other values we have x and $\pi - x$, or x and $3\pi - x$.

The total sum is $\boxed{6\pi.}$

P 25. Arcsin a + Arcsin b = Arcsin c

$c = \sin(\text{Arcsin } a + \text{Arcsin } b) = \boxed{a\sqrt{1-b^2} + b\sqrt{1-a^2}}$, where

$\cos(\text{Arcsin } a) = \sqrt{1-\sin^2(\text{Arcsin } a)} = \sqrt{1-a^2}$, etc.

D 26. We square the equations: $\sin^2 A - 2\sin A \sin B + \sin^2 B = \frac{1}{9}$

$\cos^2 A + 2\cos A \cos B + \cos^2 B = \frac{16}{9}$.

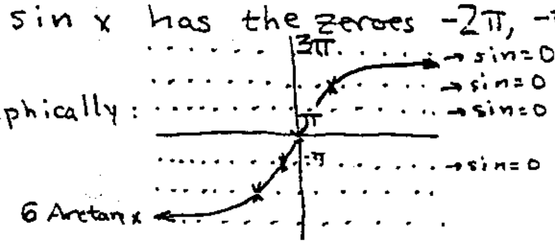
Adding, $(\sin^2 A + \cos^2 A) + 2(\cos A \cos B - \sin A \sin B)$

$+ (\sin^2 B + \cos^2 B) = \frac{17}{9}$

$2 + 2\cos(A+B) = \frac{17}{9} \Rightarrow \cos(A+B) = \boxed{-\frac{1}{18}.}$

C 27. The range of Arctan x is $(-\frac{\pi}{2}, \frac{\pi}{2})$; the range of 6Arctan x is then $(-3\pi, 3\pi)$; sin x has the zeroes $-2\pi, -\pi, 0, \pi, 2\pi$

$\Rightarrow$ $\boxed{\text{5 zeroes.}}$ Graphically:



6 Arctan x approaches, but does not reach, $\pm 3\pi$.

B 28. Since $e^{ia} = \cos a + i \sin a$, we have $e^{-ia} = \cos a - i \sin a$, so

$$\cos a = \frac{e^{ia} + e^{-ia}}{2}.$$

Similarly, $$\sin a = \frac{e^{ia} - e^{-ia}}{2i}.$$

Writing $x = \frac{\pi}{40}$ and noting that $e^{20ix} = e^{i\pi/2} = i$, we have

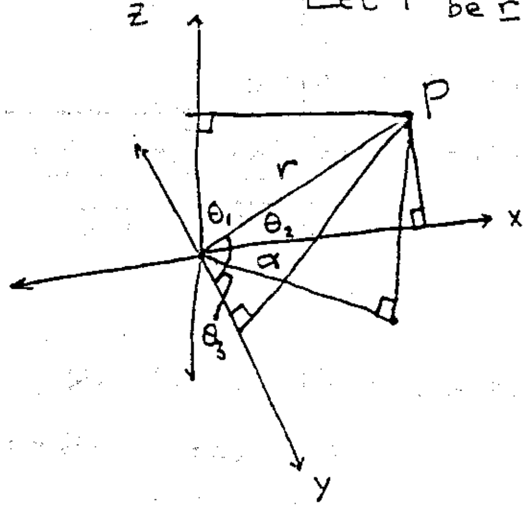
$$\sum_{n=1}^{20} \cos nx = \sum_{n=1}^{20} \frac{e^{inx} + e^{-inx}}{2} = \frac{1}{2}\left(\sum_{n=1}^{20} e^{inx} + \sum_{n=1}^{20} e^{-inx}\right).$$ Here

we have two geometric progressions, so this is

$$\frac{1}{2}\left(\frac{e^{ix}(1-e^{20ix})}{1-e^{ix}} + \frac{e^{-ix}(1-e^{-20ix})}{1-e^{-ix}}\right) = \frac{1}{2}\left(\frac{(e^{ix}+e^{-ix}-2) + i(-e^{ix}+1+e^{-ix}-1)}{2-e^{ix}-e^{-ix}}\right)$$

$$= \frac{1}{2}\left(\frac{2\cos x - 2 + 2\sin x}{2 - 2\cos x}\right) = \boxed{\frac{1}{2}\left(\frac{p+q-1}{1-p}\right).}$$

A 29.



Let P be r away from the origin.

The xyz coordinates of P are $(r\cos\theta_2, r\cos\theta_3, r\cos\theta_1)$.

Now the xyz coordinates can also be written in spherical terms, based on the angle α in the diagram (these are seen from the right triangles):

$$(r\sin\theta_1\cos\alpha,\ r\sin\theta_1\sin\alpha,\ r\cos\theta_1)$$

Thus $\cos\alpha = \dfrac{\cos\theta_2}{\sin\theta_1}$, $\sin\alpha = \dfrac{\cos\theta_3}{\sin\theta_1}$, so

$$1 = \cos^2\alpha + \sin^2\alpha = \frac{\cos^2\theta_2 + \cos^2\theta_3}{\sin^2\theta_1}$$

$$1 - \cos^2\theta_1 - \cos^2\theta_2 = \cos^2\theta_3$$

$$\cos\theta_3 = \boxed{\sqrt{1 - R^2 - S^2}} \text{ (positive).}$$

A 30. $\tan x = \dfrac{1}{\tan x} - \dfrac{1-\tan^2 x}{\tan x} = \dfrac{1}{\tan x} - 2\dfrac{1}{\tan 2x} = \cot x - 2\cot 2x.$

Using this trick, the sum telescopes:

$$\sum_{k=0}^{\infty} 2^k \tan(2^k a) = \tan a + 2\tan 2a + 4\tan 4a + \dots$$

$$= (\cot a - 2\cot 2a) + (2\cot 2a - 4\cot 4a) + (4\cot 4a - 8\cot 8a) + \dots$$

$$= \cot a$$

$$= \frac{\cos a}{\sin a} = \frac{4/5}{3/5} = \boxed{4/3}.$$