Answers to Complex Numbers

1)	D	6)	A	11)	B	16)	B	21)	A	26)	[redacted] C
2)	E	7)	D	12)	A	17)	E	22)	B	27)	D
3)	C	8)	B	13)	C	18)	C	23)	D	28)	D
4)	D	9)	E(0)	14)	C	19)	D	24)	B	29)	E(5/4)
5)	A	10)	C	15)	A	20)	E[redacted]	25)	E	30)	A

Solutions to 1993 MAΘ Complex Numbers Test

① $(5+3i)(2+i)-(4+i)(5+2i)=(7+11i)-(18+13i)=\boxed{(-11-2i)}$ Ⓓ

② The 5 fifth roots of 32 satisfy $x^5-32=0$. If the 5 roots are a, b, c, d, + e,
$(x-a)(x-b)(x-c)(x-d)(x-e)=x^5-(a+b+c+d+e)x^4+\cdots+\underline{(-abcde)}=0=x^5\underline{-32}$
$\therefore\ -abcde=-32$ (constant terms match up) $\Rightarrow [\text{product}]=abcde=\boxed{32}$ Ⓔ

③ $(1+i)^6=(\sqrt{2}\operatorname{cis}45°)^6=(\sqrt{2})^6\operatorname{cis}(45\cdot 6)=8\operatorname{cis}270°=\boxed{-8i}$ Ⓒ

④ $\underline{(1+i)(2-i)}(3+i)=\underline{(3+i)}(3+i)=\boxed{8+6i}$ Ⓓ

⑤ $a=1+\sqrt{3}i=2\operatorname{cis}60°$
$b=\operatorname{cis}15°$
$\frac{a}{b}=\frac{2\operatorname{cis}60°}{\operatorname{cis}15°}=2\operatorname{cis}(60°-15°)=2\operatorname{cis}45°=\boxed{\sqrt{2}+\sqrt{2}i}$ Ⓐ

⑥ $x^2+4x+5=(x^2+4x+4)+1=(x+2)^2+1=0 \Rightarrow x=-2\pm i$; $a=-2$ $b=\pm1$
$a^2+b^2=(-2)^2+(\pm1)^2=\boxed{5}$ Ⓐ

⑦ $(1+\sqrt{3}i)^{1/6}=(2\operatorname{cis}60°)^{1/6}=2^{1/6}\operatorname{cis}\left(\frac{60+360n}{6}\right)$ for $n=0,1,2,3,4,5$
$(n=0)$ $=2^{1/6}(\cos 10°+i\sin 10°)$ A ✓
$(n=1)$ $=2^{1/6}(\cos 70°+i\sin 70°)=2^{1/6}(\sin 20°+i\cos 20°)$ B ✓
$(n=2)$ $=2^{1/6}(\cos 130°+i\sin 130°)=2^{1/6}(-\cos 50°+i\sin 50°)$ C ✓
$(n=5)$ $=2^{1/6}(\cos 310°+i\sin 310°)=2^{1/6}(\cos 50°-i\sin 50°)$ E ✓
$(n=3)$ $=2^{1/6}(\cos 190°+i\sin 190°)=2^{1/6}(-\cos 10°-i\sin 10°)=2^{1/6}(-\sin 80°-i\sin 10°)\neq 2^{1/6}(-\sin 80°+i\sin 10°)$
Ⓓ does not satisfy

⑧ $\sum_{k=0}^{100} i^k=\underbrace{\underset{\substack{\downarrow\\1}}{i^0}+\underset{\substack{\downarrow\\i}}{i^1}+\underset{\substack{\downarrow\\-1}}{i^2}+\underset{\substack{\downarrow\\-i}}{i^3}}_{\substack{\text{every sum of 4 is 0}\\\text{(cyclic)}}}+\cdots+\underbrace{i^{96}+i^{97}+i^{98}+i^{99}}_{0}+i^{100}=i^{100}=\boxed{1}$ Ⓑ

⑨ The 3 cube roots of 8 satisfy $x^3-8=0$; if the 3 roots are a, b, + c, $(x-a)(x-b)(x-c)=$
$x^3-(a+b+c)x^2+\cdots=0=x^3-8$; Equating the x^2 term from both sides,
$-(a+b+c)=0 \Rightarrow (a+b+c)=\boxed{0}$ (E) NOTA

⑩ $\frac{\sqrt{5-12i}}{\sqrt{5+12i}} \cdot \frac{\sqrt{5-12i}}{\sqrt{5-12i}} \Rightarrow \frac{5-12i}{\sqrt{25+144}} = \frac{5}{13}, -\frac{12}{13}i \Rightarrow x=\frac{5}{13}$, $y=\frac{-12}{13}$ $\therefore x+y=\boxed{\frac{-7}{13}}$ Ⓒ

⑪ First term $a=\left(\frac{1+i}{2}\right)$; Common ratio $=\left(\frac{1+i}{2}\right)$ $\underline{\underline{\text{Sum}}}=\frac{a}{1-r}=\frac{\left(\frac{1+i}{2}\right)}{1-\left(\frac{1+i}{2}\right)}=\frac{\frac{1+i}{2}}{\frac{1-i}{2}}$

$=\frac{1+i}{1-i}=\boxed{i}$ Ⓑ

⑫ $|A|=\sqrt{(x+7)^2+x^2}=13 \Rightarrow 2x^2+14x+49=169 \Rightarrow (x+12)(x-5)=0$ $x=-12$ since $x<0$ Ⓐ

⑬ Let $z=(r,\theta)$. Then $z^2=(r^2,2\theta)$. $\frac{b}{a}=\tan\theta$ and $\tan 2\theta=\frac{3}{-4}$ from the definition of polar form; $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2(\frac{b}{a})}{1-(\frac{b}{a})^2}=\frac{-3}{4}$

$3(\frac{b}{a})^2-8(\frac{b}{a})-3=0 \Rightarrow \frac{b}{a}=3$ or $\frac{-1}{3} \Rightarrow \frac{b}{a}=\boxed{3}$ b/c $|b|>|a|$ Ⓒ

⑭ Any three points on a circle cannot be collinear. Thus, a third point cannot lie on the line which connects $(2+i)$ and $(3i)$ → $\underline{a+b=3}$ ⇒ $\boxed{(7-4i)}$ Ⓒ is on this line

⑮ $x^3=-64=64\,\text{cis}\,180° \Rightarrow x=64^{1/3}\,\text{cis}\left(\frac{180+360n}{3}\right)$ for $n=0,1,2$

$x=4\,\text{cis}\,60°, 4\,\text{cis}\,180°, 4\,\text{cis}\,300° = (2+2\sqrt{3}i), -4, (2-2\sqrt{3}i)$ Ⓐ

⑯ $(\cos 75°+i\cos 15°)(\cos 12°+i\sin 12°)(-\cos 87°+i\sin 87°)=(\text{cis}\,75°)(\text{cis}\,12°)(\text{cis}\,93°)=\text{cis}(75+12+93)=\text{cis}\,180°=\boxed{-1}$ Ⓑ

⑰ Let $x=\log_i(-1) \Rightarrow i^x=-1$; $x=\boxed{2}$ Ⓔ is a solution

⑱ The 7 7th roots satisfy $x^7+1=0$; since the x^6 coefficient is 0, the sum of all seven terms is 0. The only real root is $x=-1$, so the [sum of complex roots] = (Total) − (real) = $0-(-1)=\boxed{1}$ Ⓒ

⑲ $\left(e^{\frac{\pi}{4}i}+1\right)^4=e^{\pi i}+\binom{4}{1}e^{\frac{3\pi}{4}i}+\binom{4}{2}e^{\frac{\pi}{2}i}+\binom{4}{3}e^{\frac{\pi}{4}i}+1=-1+4\left(\frac{-\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)+6i+4\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)+1$

$=4\sqrt{2}i+6i=\boxed{(6+4\sqrt{2})i}$ Ⓓ

⑳
$$\sqrt{3+4i}+\sqrt{3-4i}=\begin{cases}\sqrt{(2+i)^2}+\sqrt{(2-i)^2}=4\\ \sqrt{(-2-i)^2}+\sqrt{(2-i)^2}=-2i\\ \sqrt{(-2-i)^2}+\sqrt{(-2+i)^2}=-4\\ \sqrt{(2+i)^2}+\sqrt{(-2+i)^2}=2i\end{cases}$$

Ⓔ <u>NOTA</u>
No ANSWERS MATCH

㉑ $z_1=3+4i$ or $4+3i$
$z_2=8+15i$ or $15+8i$ $\Rightarrow z_1+z_2=11+19i, 19+11i, 18+12i,$ or $12+18i$

$|z_1+z_2|=\sqrt{11^2+19^2}=\sqrt{482}$ Ⓐ

or

$\sqrt{18^2+12^2}=\sqrt{468}$

P2

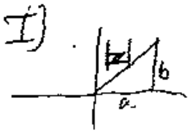
(22) If $(3+2i)$ is a solution, $(3-2i)$ is one also $\leftarrow$ its conjugate. Thus, $(x-(3+2i))(x-(3-2i))=x^2-6x+13$ is a factor

$$\begin{array}{r} x^3+x^2+8x-10 \\ x^2-6x+13\,\overline{)\,x^5-5x^4+15x^3-45x^2+164x-130} \end{array}$$

$\Rightarrow$ since the sum of the coefficients equals 0, 1 is a real solution. Factoring out $(x-1)$ leaves $x^2+2x+10$, which has complex solutions

$\therefore$ Sum of all real solutions equals $\boxed{1}$ Ⓑ

(23) I)



Triangle inequality says $|z|\le a+b$
I) false

II) $z\bar{z}=(a+bi)(a-bi)=a^2+b^2$ II) true

III) $|\bar{z}|=\sqrt{a^2+(-b)^2}=\sqrt{a^2+b^2}=|z|$ III) true Ⓓ 3 statements are true

IV) $\text{Im}(a+bi+a-bi)=\text{Im}(2a)=0$ IV) true

(24) $x-\frac{1}{x}=4i\sin9°\cos9°=2i\sin18°$. Let $x=r\,\text{cis}\,\theta$, then $x-\frac{1}{x}=r\,\text{cis}\,\theta-r^{-1}\text{cis}(-\theta)$

The real term can cancel if $r=1$, so $\text{cis}\,\theta-\text{cis}(-\theta)=2i\sin\theta=2i\sin18°\Rightarrow\theta=18°$ and

$x=\text{cis}\,18°$. $x^5=\text{cis}(18\cdot5)=\text{cis}\,90°=\boxed{i}$ Ⓑ

(25) $x=i^{i^{i^{\cdot^{\cdot}}}}\Rightarrow x=i^x\Rightarrow\ln x=x\ln i\Rightarrow\frac{\ln x}{x}=\ln i=\ln e^{\frac{\pi}{2}i}=\frac{\pi}{2}i$

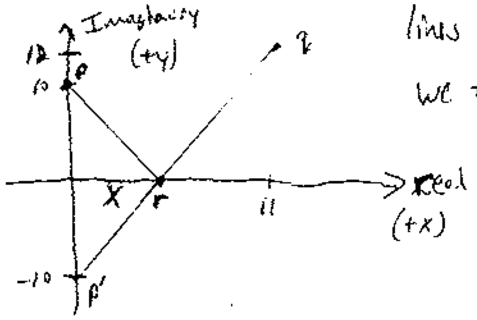
$$\frac{xi}{\ln x}=\boxed{\frac{2}{\pi}}$$ Ⓔ

(26) $(a+bi)=(\text{cis}\,30°+\text{cis}\,60°)^6=\text{cis}\,180°+\binom{6}{1}\text{cis}(30\cdot5+60)+\binom{6}{2}\text{cis}(30\cdot4+60\cdot2)+\binom{6}{3}\text{cis}(30\cdot3+60\cdot3)$
$+\binom{6}{4}\text{cis}(30\cdot2+60\cdot4)+\binom{6}{5}\text{cis}(30+60\cdot5)+\text{cis}(360°)=-1+6\,\text{cis}\,210°+15\,\text{cis}\,240°+20\,\text{cis}\,270°+15\,\text{cis}\,300°$
$+6\,\text{cis}\,330°+1=6(\text{cis}\,210°+\text{cis}\,330°)+15(\text{cis}\,240°+\text{cis}\,300°)-20i=6(\frac{1}{2}+\frac{1}{2})i+15(-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2})i-20i$
$=-6i-15\sqrt{3}i-20i=\boxed{(-15\sqrt{3}-26)i}$ ~~Ⓒ~~ A

(27) $x^2+x+\frac{1}{x}=-1\xRightarrow[\text{we can multiply by }x]{\text{since }x\neq0,}x^3+x^2+1=-x\Rightarrow x^3+x^2+x+1=\frac{x^4-1}{x-1}=0$; thus, x is a 4th root of 1 that isn't $(x=1)$

$$x^8+\frac{1}{x^8}=(x^4)^2+\frac{1}{(x^4)^2}=1^2+\frac{1}{1^2}=\boxed{2}$$ Ⓓ

(28) Plot these points graphically, What we seek is to minimize is the length of the lines $\overrightarrow{pr}$ and $\overrightarrow{rq}$. If we reflect p across the real axis to p', we find that $|\overrightarrow{pr}| = |\overrightarrow{p'r}|$. The minimum length of $\overrightarrow{p'r} + \overrightarrow{rq}$ occurs when the three points are collinear. The equation of the line $p'q$ is $y = 2x - 10$. Its x-intercept is $\boxed{X=5}$ Ⓓ, which is what we seek.



(29) $\cos x = \frac{e^{xi}+e^{-xi}}{2} \Rightarrow \cos(\ln 2^i) = \frac{e^{i\ln 2^i}+e^{-i\ln 2^i}}{2} = \frac{e^{\ln 2^{-1}}+e^{\ln 2}}{2}$

$= \frac{2^{-1}+2}{2} = \boxed{\frac{5}{4}}$ Ⓔ

(30) $\ln\left[-(i\cos 2+\sin 2)^2\right] = \ln\left[-(\sin^2 2-\cos^2 2+2i\sin 2\cos 2)\right] = \ln\left[\cos^2 2-\sin^2 2-2i\sin 2\cos 2\right]$

$= \ln\left[\cos 4 - i\sin 4\right] = \ln(\text{cis}-4) = \ln(e^{-4i}) = \boxed{-4i}$ Ⓐ