Alpha Division

1. Solve for number base x if x > 0 and if $(53)_{\text{twelve}} = (333)_x$

$$5(12) + 3 = 3x^2 + 3x + 3$$
$$3x^2 + 3x - 60 = 0$$
$$x^2 + x - 20 = 0$$
$$(x+5)(x-4) = 0$$
$$x = -5 \text{ or } x = 4$$

$$\boxed{4}$$

2. 3 angles of a convex hexagon are each 100°, and 2 others are each 130°. What is the degree measure of the other angle?

The total measure of the angles of a hexagon is $180°(6-2) = 720°$

The five angles we are given total $3(100)° + 2(130)° = 560°$
The remaining angle is $720 - 560 = \boxed{160°}$

3. Solve for x: $\log_6(x+1) = 2\log_6(x-1)$

$2\log_6(x-1) = \log_6(x-1)^2$, so
$\log_6(x+1) = \log_6(x-1)^2$

$\therefore \quad x+1 = (x-1)^2$
$$x+1 = x^2 - 2x + 1$$
$$x^2 - 3x = 0$$
$$x = 0 \quad \text{or} \quad x = 3$$

Since the log of zero is undefined, $\boxed{x = 3}$

4. Three women, Anne, Elizabeth, and Linda, and their husbands (not necessarily respectively), Barry, Randy, and Nick go to the market to buy horses. Each person spent as many dollars per horse as the number of horses that person bought. Each woman spent \$63 more than her husband. Anne bought 23 horses more than Randy, and Elizabeth bought 11 more than Barry. Who is Linda's husband?

Each person spends a square number of dollars. So each husband-wife pair can be denoted by (x,y) where the wife spends x^2, the husband spends y^2, and $x^2-y^2=63$. The possible values of (x,y) are $(32,31)$, $(12,9)$, and $(8,1)$. Anne must have bought 32 horses, Randy 9, Elizabeth 12, and Barry 1. Linda must have bought 8 horses + be married to $\boxed{\text{Barry.}}$

5. A homeowner is charged \$1 for every full degree the low temperature on a given day falls below 40° (e.g. no charge if the low is 39.2°, but a \$1.00 charge if the low is 38.9°). If the low temperature in December is described by $55-\frac{3}{4}D$, where D is the day of the month, what is the homeowner's bill for December?

The amount the temperature is below 40 is $40-(55-\frac{3}{4}D)$

$=\frac{3}{4}D-15=\frac{3}{4}(D-20)$. This exceeds 1 when $D\geq 22$.

We want to find $\sum_{i=22}^{31}\left[\frac{3}{4}(D-20)\right]=\left[1\frac{1}{2}\right]+\left[2\frac{1}{4}\right]+\ldots+\left[8\frac{1}{4}\right]$

$=1+2+3+3+4+5+6+6+7+8$

$=\boxed{45}$

6. What is the remainder when 5^{1993} is divided by 7 ?

5^1	R5
5^2	R4
5^3	R6
5^4	R2
5^5	R3
5^6	R1

5^6 leaves a remainder of 1 when divided by 7, and so does $(5^6)^{332}=5^{1992}$.

$5^{1993}=5\cdot 5^{1992}$ leaves a remainder of $\boxed{5.}$

7. Find the largest prime divisor of $57^2 \cdot 46^3$.

$57^2 \cdot 46^3 = (3 \cdot 19)^2 \cdot (2 \cdot 23)^3$

The largest prime factor is $\boxed{23.}$

8. If $x^2+y^2=101$ and $x+y=11$, find $x-y$ if $x-y>0$.

If $x+y=11$, $(x+y)^2=121$, so

$$\begin{array}{r} x^2+2xy+y^2=121 \\ x^2 \quad\quad\; +y^2=101 \\ \hline 2xy \quad\quad = 20 \end{array}$$

$x^2+2xy+y^2-4xy=81$

$x^2-2xy+y^2=81$

$(x-y)^2=81$

$\therefore (x-y)=9$ if $(x-y)>0$

$\boxed{9}$

9. Find the maximum distance from (7, -9) to any point on the circle $x^2-4x+y^2-6y-12=0$.

$x^2-4x+y^2-6y-12=0$

$(x^2-4x+4)+(y^2-6y+9)=25$

$(x-2)^2+(y-3)^2=25$

$\therefore$ The circle has center (2, 3) and radius 5.

The maximum distance from the point to the circle is the sum of the distance from the point to the center and the circle's radius.

The distance from (2,3) to (7,-9) is 13.

$13+5=\boxed{18}$

10. If $f(x)=px+q$, $f(f(2))=-2$, and $f(f(3))=7$, find the two possible values for q.

$f(f(x)) = f(px+q) = p(px+q)+q = p^2x+pq+q.$

Since this is linear, $(2,-2)$ and $(3,7)$ determine $y=9x-20$.

$p^2=9$, so $p=3$ or $p=-3$.

$\therefore q=-5$ or $q=10$

$\boxed{\begin{array}{c}-5\\10\end{array}}$ (must have both answers)

11. Find the smallest positive integer n such that n/2 is a perfect square and n/3 is a perfect cube.

$n = 2^a 3^b$ so $\frac{n}{2} = 2^{a-1} 3^b$ and $\frac{n}{3} = 2^a 3^{b-1}$

To be a perfect square, a−1 and b must both be even. Similarly, a and b−1 are divisible by 3. The smallest such are a = 3 and b = 4.

$2^3 \cdot 3^4 = \boxed{648}$

12. Solve for y ($0° \le y \le 90°$): $\cot^2 y = 3$.

$\frac{\cos^2 y}{\sin^2 y} = 3$

$\cos^2 y = 3 \sin^2 y$

$1 - \sin^2 y = 3 \sin^2 y$

$1 = 4 \sin^2 y$

$\sin^2 y = \frac{1}{4}$

$\sin y = \pm \frac{1}{2}$

$\boxed{y = 30°}$

or it can be done as:

$\cot y = \sqrt{3}$ if $0 \le y \le 90$

so $\boxed{y = 30°}$

13. A jar contains 12 red balls and 6 white balls. One ball is randomly drawn from the jar and is replaced by one of the opposite color. If a ball is now randomly drawn from the jar, what is the probability that it is red?

There is a $\frac{12}{18} = \frac{2}{3}$ chance that a red ball will be replaced, leaving 11R + 7W.

There is a $\frac{6}{18} = \frac{1}{3}$ chance that a white ball will be replaced, leaving 13R + 5W.

$\frac{2}{3} \cdot \frac{11}{18} + \frac{1}{3} \cdot \frac{13}{18} = \boxed{\frac{35}{54}}$

14. A cone of height 6 is cut along a plane parallel to its base so that the volume of the smaller cone is 1/3 that of the original cone. What is the height of the smaller cone?
(Be certain to indicate your answer as a simplified radical.)

$$\frac{V_1}{V_2} = \frac{1}{3}$$

$$\left(\frac{h_1}{h_2}\right)^3 = \frac{(h_1)^3}{6^3} = \frac{1}{3}$$

$$3(h_1)^3 = 6^3$$

$$h_1 = \boxed{2\sqrt[3]{9}}$$

15. In a single elimination tournament with 16 teams in the first round, teams are paired and matches are held. In the second round, winners from the first round are paired similarly, and so on through every round, until two teams play a final match for the championship. Once the championship is determined, all teams beaten directly by the champion then enter a single elimination tournament run in the same manner as used to determine the champion, in order to determine the second-best team. Starting from the beginning of the 16 team tournament, how many matches are required to determine the second-best team?

To determine the best team requires 15 matches - each match eliminates one team. The second best team could be any of the teams defeated by the best team (the two best teams might meet in the first round, for example). Those four teams must play another tournament, requiring 3 more matches. $15 + 3 = \boxed{18.}$