Mathematics Invitational
Plant City High School
February 22, 1992
Calculus Team
Answers

1. $3/2 + \tan 1$

2. 5

3. 154

4. $(1+\sqrt{3})/2$

5. $2/\pi$

6. $-3\sqrt{10}/10$

7. $(1/2)\arctan 2(x-1) + c$

8. $-43/3$

9. $\ln 2$

10. -231

11. $35/6 = 5.8\overline{3}$

12. $\sqrt{3}/2 - 1/2$

13. $12/625\ln 5$

14. 78

15. $1/e$

Calculus Solutions
Team 1992 PCHS

1. $A(x) = (\sin x)(\cos x)$ $\quad B(x) = (\sec^2 x)(\cos^2 x)$

$= \frac{1}{2}\sin 2x \qquad = \frac{\cos^2 x}{\cos^2 x}$

$A'(x) = \frac{1}{2}\cdot 2\cos 2x \qquad = 1$

$= \cos 2x \qquad B'(x) = 0$

$A'(\frac{\pi}{3}) = \cos \frac{2\pi}{3} \qquad B'(\frac{\pi}{3}) = 0$

$= -\frac{1}{2}$

$C(x) = \operatorname{Arcsin} 2x \qquad D(x) = x^{\tan x}$

$u = 2x \quad du = 2dx \qquad \ln(D(x)) = \tan x \ln x$

$u^2 = 4x^2 \quad \frac{2}{\sqrt{1-4x^2}} \qquad \frac{1}{D(x)}\cdot D'(x) = (\sec^2 x)\ln x + \frac{\tan x}{x}$

$= 2/\sqrt{1-4x^2} \qquad \frac{1}{D(1)}\cdot D'(1) = (\sec^2 1)(\ln 1) + \frac{\tan 1}{1}$

$C'(0) = 2/\sqrt{1-4(0)} \qquad \frac{1}{D(1)}\cdot D'(1) = 0 + \tan 1$

$= 2/1 \qquad D'(x) = \tan 1 \cdot D(1)$

$= \tan 1 \cdot 1^{\tan 1}$

$A'(\frac{\pi}{3}) + B'(\frac{\pi}{3}) + C'(0) + D'(1) \qquad = \tan 1$

$-\frac{1}{2} + 0 + 2 + \tan 1$

$\boxed{3/2 + \tan 1}$

2. $2x^2 - x^2y^2 + y^{-2} - x = 7$

$4x - [2xy^2 + 2x^2y\frac{dy}{dx}] + -2y^{-3}\frac{dy}{dx} - 1 = 0 \qquad (-2, 1)$

$4(-2) - [-4\cdot 1 + 2\cdot 4\cdot 1\frac{dy}{dx}] - 2\frac{dy}{dx} = 1$

$-8 - [-4 + 8\frac{dy}{dx}] - 2\frac{dy}{dx} = 1 \qquad 2x + 2y = 7$

$-10\frac{dy}{dx} = 5 \qquad 2y = -2x + 7$

$\frac{dy}{dx} = -1/2 \qquad y = -1x + 7/2$

$g(x) = 2x^2 - 3x + 4 \qquad g(1/2) = 2(\frac{1}{4}) - 3(\frac{1}{2}) + 4$

$g'(x) = 4x - 3 \qquad = 1/2 - 3/2 + 8/2$

$4x - 3 = -1 \qquad A = -1/2 \qquad = 6/2$

$4x = 2 \qquad (B, C) \qquad = 3$

$x = 1/2 \qquad (1/2, 3) \qquad \frac{A+C}{B} = \frac{-\frac{1}{2}+\frac{6}{2}}{1/2}$

$= 5/2 \cdot \frac{2}{1} = \boxed{5}$

3. $A(x) = \frac{\cancel{(x-3)}(x^2+3x+9)}{\cancel{(x-3)}} = \lim_{x\to 3} x^2 + 3x + 9$

$= 9 + 9 + 9$

$\boxed{= 27}$

$B(x) = \lim_{x\to 0} \frac{4\sin 4x}{4x} = \lim_{x\to 0} 4\left[\frac{\sin 4x}{4x}\right] = 4\cdot 1$

$= \boxed{4}$

$C(x) = \lim_{x\to\infty} \frac{\frac{3x}{x}}{\frac{x}{x}+\frac{2}{x}} - \frac{\frac{x}{x}-\frac{1}{x}}{\frac{2x}{x}-\frac{1}{x}} = \lim_{x\to\infty} \frac{3}{1} - \frac{1}{2}$

$3 - 1/2 = \boxed{\frac{5}{2}}$

$D(x) = \lim_{x\to 2} \frac{8 + 3(4) - 4}{16 + 3(4) - 1} = \boxed{\frac{16}{3}}$

$\frac{AD + BC}{}$

$(27)(\frac{16}{3}) + 4(\frac{5}{2})$

$144 + 10 = \boxed{154}$

4. $f(x) = \sin x - \cos x \qquad f^{147}(x) = f$

$f'(x) = \cos x + \sin x \qquad \frac{147}{4} = 36\ 3/4$

$f''(x) = -\sin x + \cos x \qquad R\ 3$

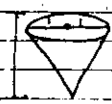
$f'''(x) = -\cos x - \sin x \leftrightarrow$ hence $f^{147}(x) = f'''(x)$

$f^{iv}(x) = \sin x - \cos x \qquad f^{147}(\frac{4\pi}{3}) = -\cos\frac{4\pi}{3} - \sin\frac{4\pi}{3}$

$f^{v}(x) = \cos x + \sin x \qquad -(-\frac{1}{2}) - (\frac{-\sqrt{3}}{2})$

$\boxed{1/2 + \sqrt{3}/2}$

5.



10

$\frac{dV}{dt} = {}^{+}5\,m^3/\mathrm{sec} \qquad V = 1/3\,\pi R^2 h$

$h = 5 \quad$ ~~have~~ $r = 1/2 \qquad = 1/3\,\pi r^2 (10r)$

$= \frac{10}{3}\pi r^3$

$\frac{r = 1}{h = 10} \quad \frac{r}{h} = \frac{1}{10} \quad 10r = h \qquad \frac{dV}{dt} = 10\pi r^2 \frac{dr}{dt}$

${}^{+}5 = 10\pi(\frac{1}{2})^2\frac{dr}{dt}$

$\boxed{\frac{2}{\pi}} = \frac{5\cdot 4}{10\pi} = \frac{dr}{dt}$

6. $A = \lim_{x\to 2} \frac{\sqrt{2x+1} - \sqrt{5}}{x-2}$

$\frac{\sqrt{2(2)+1} - \sqrt{5}}{2-2} = \frac{5-5}{0} = \frac{0}{0}$

hence L'Ho Rule: $\frac{1/2(2x+1)^{-1/2}(2)}{1} = \frac{2}{2\sqrt{2x+1}} = \frac{1}{\sqrt{2(2)+1}} = \frac{1}{\sqrt{5}}$

$B = \lim_{x\to -\infty} \frac{3x+4}{\sqrt{2x^2+5}} = \frac{\frac{3x}{-x}+\frac{4}{-x}}{\sqrt{\frac{2x^2}{x^2}+\frac{5}{x^2}}} = \frac{-3}{\sqrt{2}}$

$C =$ let $y = \lim_{x\to 0^+} x^x$

$\ln y = \lim_{x\to 0^+} x\ln x$

$\ln y = \lim_{x\to 0^+} \frac{\ln x}{1/x}$

Using L'Hopital's Rule

$\ln y = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \frac{1}{x}\cdot\frac{-x^2}{1} = -x$

$\ln y = \lim_{x\to 0^+} x$

$\ln y = 0$

$e^0 = y$

$1 = y$

$D.\ \lim_{x\to 0} \frac{e^0 - e^{-0} - 2(0)}{0 - \sin 0}$

$\frac{1-1-0}{0-0} = \frac{0}{0}$

$\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}$

$\lim_{x\to 0} \frac{e^0 + e^0 - 2}{1 - \cos 0}$

$\frac{1+1-2}{1-1} = \frac{0}{0}$

$\lim_{x\to 0} \frac{e^x - e^{-x}}{+\sin x} = \frac{e^0 - e^0}{\sin 0}$

$= \frac{1-1}{0} = \frac{0}{0}$

$\lim_{x\to 0} \frac{e^x + e^x}{\cos x} = \frac{e^0 + e^0}{\cos 0}$

$= \frac{1+1}{1} = \frac{2}{1}$

$A = 1/\sqrt{5} \quad B = -3/\sqrt{2}$

$C = 1 \quad D = 2$

$AB_{\,}(D - C)$

$\frac{1}{\sqrt{5}}\cdot\frac{-3}{\sqrt{2}}(2-1)$

$\frac{-3}{\sqrt{10}}(1) = \boxed{\frac{-3\sqrt{10}}{10}}$

7. $\int \frac{dx}{4x^2 - 8x + 5} \qquad \frac{4(x^2 - 2x + 1) - 4 + 5}{4(x-1)^2 + 1}$

$\frac{1}{2}\int \frac{du}{u^2 + 9} \qquad u = 2(x-1) \quad a = 1 \qquad du = 2dx$

$u^2 = 4(x-1)^2 \quad a^2 = 1$

$\boxed{\frac{1}{2}\arctan\frac{2(x-1)}{1} + C}$

8. $f(x) = x|x-3|$

$f'(1) = |x-3| + \frac{x(x-3)(1)}{|x-3|}$

$|1-3| + \frac{(1)(1-3)(1)}{|(1-3)|}$

$|-2| + \frac{(-2)}{|(-2)|}$

$2 + \frac{(-2)}{2} = 2 + -1 = 1$

$f'(1) = 1$

$\int_{-2}^{2} x|x-3| =$

$\int_{-2}^{0} -(x)(x-3) + \int_{0}^{2} -(x)(x-3)$

$-\int_{-2}^{0} x^2 - 3x \qquad \int_0^2 -x^2 + 3x$

$-\left[\frac{1}{3}x^3 - \frac{3}{2}x^2\Big|_{-2}^{0}\right] + -\frac{1}{3}x^3 + \frac{3}{2}x^2\Big|_0^2$

$-[(0-0) - (-\frac{8}{3} - 6)] + -\frac{8}{3} + 6 - 0$

$-[\frac{8}{3} + 6]$

$\frac{-26}{3} + \frac{10}{3}$

$-16/3$

$\lim_{x \to -2} -2|(-2-3)| = -2|(-5)| = -10$

$A = a + c$

$1 + \frac{-16}{3} + -10$

$1 + \frac{-46}{3}$

$\boxed{-43/3}$

9. $\int_4^9 \frac{dx}{x - \sqrt{x}}$

$\int_4^9 \frac{dx}{\sqrt{x}(\sqrt{x}-1)}$

Let $u = \sqrt{x} - 1$

$du = \frac{1}{2}x^{-1/2}dx$

$2du = \frac{dx}{\sqrt{x}}$

$2\int_1^2 \frac{du}{u} = 2\ln u \Big|_1^2$

$2\ln 2 - 2\ln 1$

$2\ln 2 - 0$

$\boxed{2\ln 2 = A}$

$B(x) = y$

$B(x) = y$

$\ln y = (x^2+1)\ln x$

$\frac{1}{y}\frac{dy}{dx} = 2x\ln x + \frac{x^2+1}{x}$

$B'(1) =$

$\frac{1}{y}\frac{dy}{dx} = 2\ln 1 + 2$

$\frac{1}{y}\frac{dy}{dx} = 2$

$\frac{dy}{dx} = 2x^{x^2+1}$

$= 2 \cdot 1$

$\boxed{B = 2}$

$\frac{A}{B'(1)} = \frac{2\ln 2}{2}$

$= \ln 2$

10. $f(x) = x^3 - 2x^2 - 15x + 3$

$f'(x) = 3x^2 - 4x - 15$

$g(x) = \int_0^x \frac{3t}{2}\,dt = \frac{3}{2}\cdot\frac{1}{2}t^2\Big|_0^x = \frac{3x^2}{4}$

$A = -245$

$B = 0$

$C = 14$

$g(x) = \frac{3x^2}{4}$

$C = \int_2^4 \frac{3x^2}{4} \cdot \frac{3}{4\cdot 3}x^3\Big|_2^4$

$g(2) = \frac{3\cdot 4}{4} = 3$

$= \frac{x^3}{4}\Big|_2^4$

$f'(3) = 3\cdot 9 - 4(3) - 15$

$= 27 - 12 - 15$

$= 0$

$\frac{64}{4} - \frac{8}{4}$

$16 - 2 = 14$

12 $\int_1^2 x^3 - 2x^2 - 15x + 3$

$\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{15}{2}x^2 + 3x\Big|_1^2$

$4 + B + C$

$-245 + 0 + 14$

$\boxed{-231}$

$\frac{360}{45}$

$3x^4 - 8x^3 - 90x^2 + 36x\Big|_1^2$

$48 - 64 - 360 + 72 - (3 - 8 - 0 + 36)$

$-304 + 59$

-245

-88
$+39$
-79

-360
$+72$
-288
48
-240
-64
-304
59
-245

11. $f(x) = x^{1/3} \qquad g(x) = x^{1/4}$

$x = 27 \qquad x = 16$

$dx = 3 \qquad dx = 4$

$f(x + \Delta x) = f(x) + f'(x)dx$

$(27)^{1/3} + \frac{1}{3}(27)^{-2/3} \cdot 3$

$3 + 3^{3 \cdot -2/3}$

$3 + 1/9$

$3\frac{1}{9}$

$\frac{28}{9}$

$\frac{28}{9} \cdot \frac{15}{8} = \boxed{\frac{35}{6}}$

$g(x + \Delta x) = g(x) + g'(x)dx$

$= 16^{1/4} + \frac{1}{4}(16)^{-3/4} \cdot 4$

$= 2 - 2^{4 \cdot -3/4}$

$= 2 - 1/8$

$= 1\frac{7}{8}$

$= 15/8$

12. Taylor Series for

$\int_{\pi/4}^{\pi/3} \sin x + \cos x \, dx$

$-\cos x + \sin x \Big|_{\pi/4}^{\pi/3}$

$-\cos\frac{\pi}{3} + \sin\frac{\pi}{3} - \left(-\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)$

$-\frac{1}{2} + \frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)$

$\boxed{\frac{\sqrt{3}}{2} - \frac{1}{2}}$

13. $\int_1^2 5^{-2x}\,dx$ let $u = -2x$

$du = -2dx$

$-\frac{1}{2}du = dx$

$-\frac{1}{2}\int_{-2}^{-4} 5^u\,du$

$-\frac{1}{2}\cdot\frac{1}{\ln 5}5^u = -\frac{1}{2\ln 5}5^{-2x}\Big|_1^2 = \frac{-1}{2\ln 5}\left[\frac{1}{625} - \frac{1}{25}\right]$

$\frac{-1}{2\ln 5} \cdot \frac{-24}{625}$

$\frac{12}{\ln 5 (625)}$

14. $\int_0^3\int_0^{5-x} 4y\,dy\,dx$

$\int \frac{4}{2}y^2\Big|_0^{5-x} dx$

$\int_0^3 2y^2\Big|_0^{5-x} dx$

$\int_0^3 [2(5-x)^2 - 0]\,dx$

$2\int_0^3 (5-x)^2\,dx$

let $u = 5-x$, $du = -dx$

$-2\int u^2\,du = -2\cdot\frac{1}{3}u^3$

$= -\frac{2}{3}(5-x)^3\Big|_0^3$

$-\frac{16}{3} - (-\frac{250}{3})$

$\frac{250-16}{3} = \frac{234}{3} = \boxed{78}$

15. $\int_0^1 \frac{e^x - 1}{e^x}\,dx$

$\int_0^1 \frac{e^x}{e^x} - \frac{1}{e^x}\,dx$

$\int_0^1 1 - e^{-x}\,dx$

$x + e^{-x}\Big|_0^1$

$1 + e^{-1} - (0 + e^0)$

$1 + \frac{1}{e} - 0 - 1$

$\boxed{1/e}$