Answers to Calculus Individual and Team February 24, 1996

Individual	***Team***
1. A	1. $x - 15y + 1 = 0$
2. B	2. $\frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C$
3. D	3. 15
4. C	4. $(-\infty, -1) \cup (1,2)$
5. A	5. $e^{3/4}$ or e^{75}
6. D	6. 8/9
7. E	7. $336\pi^2$
8. D	8. 64/9
9. B	9. 1/4
10. C	10. 843/1393
11. B	11. 4
12. A	12. 1024/3
13. D	13. - 8.85 or - 177/20
14. B	14. 4
15. C	15. $\pi/4$
16. A	
17. C	
18. C	
19. E	
20. E	
21. B	
22. E	
23. B	
24. D	
25. A	
26. C	
27. C	
28. C	
29. D	
30. E	

Calculus Individual Solutions

February 24, 1996

1.
$$y=\frac{3(x-2)\cancel{(x+1)}}{\cancel{(x+1)}(x+3)(x+5)}$$
$x=-3, x=-5$ are vertical asy.

$\lim_{x\to\infty} f(x)=0$. So $y=0$ is asy.

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2. $y=x^{\frac{4}{3}}+x^{\frac{1}{3}}$
$y'=\frac{4}{3}x^{\frac{1}{3}}+\frac{1}{3}x^{-\frac{2}{3}}$
$y''=\frac{4}{9}x^{-\frac{2}{3}}-\frac{2}{9}x^{-\frac{5}{3}} \rightarrow \frac{2}{9}x^{-\frac{5}{3}}(2x-1)=0$

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0 $\frac{1}{2}$

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3. A cusp occurs at $x=5$, so $f(x)$ is not differentiable at $x=5$.

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4. $a(t)=3t^2-18t+26$
$v(t)=t^3-9t^2+26t-24$
$s(t)=\frac{1}{4}t^4-3t^3+13t^2-24t+2$
$s(6)=2$

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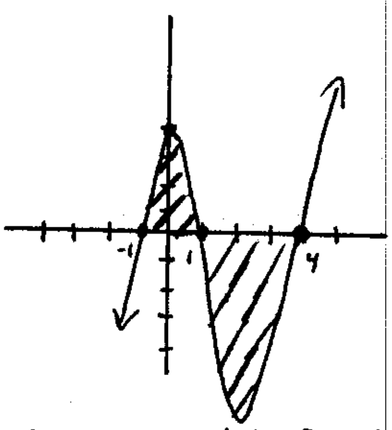
5.



$$\int_{-1}^{1} x^3-4x^2-x+4\,dx+\int_{4}^{1} x^3-4x^2-x+4\,dx=\frac{253}{12}$$

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$y=x^2(1-x)^{\frac{1}{2}}$
$y'=2x(1-x)^{\frac{1}{2}}-\frac{1}{2}x^2(1-x)^{-\frac{1}{2}}$
$y''=2(1-x)^{\frac{1}{2}}-x(1-x)^{-\frac{1}{2}}-x(1-x)^{-\frac{1}{2}}-\frac{1}{4}x^2(1-x)^{-\frac{3}{2}}$
$f''(-4)\approx 7.692$

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7. $f'(x)=-6x+1=\frac{-3-(-26)}{2-3}$
$-6x+1=-23$
$x=4$
$f(4)=-49$

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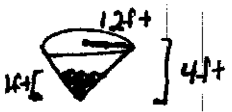
8.



$r=3h$
$\frac{dV}{dt}=4\text{ ft}^3/\text{min}$

$V=\frac{\pi}{3}r^2h$
$V=\frac{\pi}{3}(3h)^2h$
$V=\frac{\pi}{3}(9h^3)$
$V=3\pi h^3 \rightarrow \frac{dV}{dt}=9\pi h^2\frac{dh}{dt}$
$4=9\pi(1)^2\frac{dh}{dt}$
$\frac{dh}{dt}=\frac{4}{9\pi}$

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9. $\ln 5\approx 1.609$
$\ln 25=2\ln 5\approx 3.218$
$y=\ln x$
$dy=\frac{1}{x}dx \rightarrow dy=\frac{1}{25}(.15)$
$dy=.006$
$\ln(25.15)\approx 3.218+.006=3.224$

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10. $f(x)=(\sin x)^x$
$\ln y=\ln(\sin x)^x$
$\ln y=x\ln\sin x$
$\frac{1}{y}\cdot y'=\ln\sin x+\frac{x\cos x}{\sin x}$
$y'=(\sin x\ln\sin x+x\cos x)(\sin x)^{x-1}$

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11.

$W=\pi r^2 d\int_0^h h-x\,dx$
$W=42\pi(1)^2\int_0^8 8-x\,dx$
$W=1344\pi\approx 4000$

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12. $\frac{dy}{dx}=2xy$
$\int\frac{dy}{y}=\int 2x\,dx$
$\ln y=x^2+C \rightarrow 0=4+C$
$C=-4$
$\ln y=16-4$
$\ln y=12$
$y=e^{12}$

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13. $\frac{dT}{dt} = k(T - T_0)$

$\int \frac{dT}{T-T_0} = \int k\,dt$

$\ln|T - T_0| = kt + C$

$\ln|290 - 20| = 0(k) + C$

$C = \ln 270$

$\ln|110 - 20| = 1(k) + \ln 270$

$\ln 90 = k + \ln 270$

$k = \ln \frac{1}{3}$

$\ln|T - 20| = 2(\ln \frac{1}{3}) + \ln 270$

$\ln|T - 20| = \ln 30$

$T - 20 = 30$

$T = 50°$

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14. $\frac{d(x^2-2x+1)}{d(2x)} = \frac{\frac{d}{dx}(x^2-2x+1)}{\frac{d}{dx}(2x)} = \frac{2x-2}{2} = x - 1$

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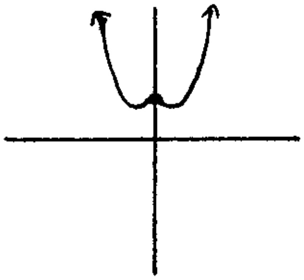
5.



I. $y = x^2 + 2e^{-x^2}$

$y' = 2x - 4xe^{-x^2}$

$y' = 2x(1 - 2e^{-x^2}) = 0$

$x = 0,\ 1 - 2e^{-x^2} = 0$

$e^{-x^2} = \frac{1}{2}$ $-x^2 = \ln\frac{1}{2}$

$x^2 = \ln 2$ $x = \pm\sqrt{\ln 2}$

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$-\sqrt{\ln 2}$ 0 $\sqrt{\ln 2}$

min. value at $x = \pm\sqrt{\ln 2}$

$f(\sqrt{\ln 2}) = \ln 2 + 2e^{-\ln 2}$

$= \ln 2 + 2(e^{\ln \frac{1}{2}})$

$= \ln 2 + 2(\frac{1}{2})$

$= \ln 2 + 1 = \ln 2 + \ln e = \ln(2e)$

I is true

II is false because there are 3 critical points

$((0, 2)\ (\sqrt{\ln 2}, \ln(2e))\ (-\sqrt{\ln 2}, \ln(2e)))$

III is false by inspection of the graph

IV is also false by inspection of the graph

So, the are 3 that are false

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16. $\lim_{x\to\infty} x - \sqrt{x^2 - 7x}$ Let $x = \frac{1}{h}$

$\lim_{h\to 0} \frac{1}{h} - \sqrt{\frac{1}{h^2} - \frac{7}{h}}$

$\lim_{h\to 0} \frac{1}{h} - \sqrt{\frac{1-7h}{h^2}}$

$\lim_{h\to 0} \frac{1}{h} - \frac{\sqrt{1-7h}}{h}$

$\lim_{h\to 0} \frac{1 - \sqrt{1-7h}}{h}$ - L'Hopital's rule

$\lim_{h\to 0} \frac{7}{2}(1-7h)^{-\frac{1}{2}} = \frac{7}{2}$

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17. Speed increases wherever $v(t)$ and $a(t)$ have the same sign.

$s(t) = t^3 - t^2 + 4$

$v(t) = 3t^2 - 2t$ $a(t) = 6t - 2$

$t(3t - 2) = 0$ $6t - 2 = 0$

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0 $\frac{2}{3}$ $\frac{1}{3}$

Same sign on $(0, \frac{1}{3}) \cup (\frac{2}{3}, \infty)$

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18. $\int_0^{\frac{\pi}{2}} 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \ldots dx = \int_0^{\frac{\pi}{2}} \cos x\,dx$

$\sin x \Big]_0^{\frac{\pi}{2}} = 1$

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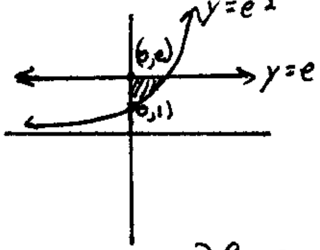
19.



$A = \int_1^e 2\ln y\,dy = 2y\ln y - 2y \Big]_1^e = 2e - 2e + 2 = 2$

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20. I. False. Not defined at $x = -2$

II. False. Not continuous at $x=-2$, $x=1$

III. False. Average Value does exist.

$$A.V. = \frac{1}{5-(-3)}\left[\int_{-3}^{-2} x^2+2\,dx + \int_{-2}^{1}\frac{x^2-4}{x+2}\,dx + \int_{1}^{5}\frac{x^2-x^3}{x-1}\,dx\right]$$

IV. $\lim_{x\to 1^-} = -1$. $\lim_{x\to 1^+} = -1$

IV is the only true statement

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21. $\frac{x^2}{x-1} - \left(\frac{x^4}{x^2-1}\right)2x$

$$\frac{x^2}{x-1} - \frac{2x^5}{x^2-1} = \frac{x^3+x^2-2x^5}{x^2-1} = \frac{x^3(1+\frac{1}{x}-2x^2)}{x^2-1}$$

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22. $A.V. = \frac{1}{5-1}\int_1^5 1+x^{-2}\,dx$

$$\frac{1}{4}\left(x - \frac{1}{x}\Big]_1^5\right) = \left(\frac{\overset{6}{\cancel{24}}}{5}\right)\frac{1}{\cancel{4}} = \frac{6}{5}$$

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23. $f(x) = 2^{x^2}$

$f'(x) = 2x(2^{x^2})(\ln 2)$

$f'(2) = 4(16)(\ln 2) = 64\ln 2$

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24. $\lim_{x\to 2}\frac{x^3-2x-4}{x-2}$ - L'Hopitals

$\lim_{x\to 2}\frac{3x^2-2}{1} = 10$

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25. $\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is an odd function. $5x^5-2x^3+x$ is odd, so $\int_{-a}^{a} 5x^5-2x^3+x = 0$. The only necessary part is $\int_{-a}^{a} -4\,dx = 3$

$-4x\Big]_{-a}^{a} = 3$. $-8a = 3$. $a = -\frac{3}{8}$

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26. Arc length $= \int_b^a \sqrt{1+(f'(x))^2}\,dx$
from $x=b$ to $x=a$

$f(x) = 2x^{\frac{3}{2}}$ $b = 0$

$f'(x) = 3\sqrt{x}$ $a = \frac{5}{3}$

$$A.L. = \int_0^{\frac{5}{3}} \sqrt{1+9x}\,dx$$

$$AL = \frac{2}{27}(1+9x)^{\frac{3}{2}}\Big]_0^{\frac{5}{3}} = \frac{2}{27}(64) - \frac{2}{27} = \frac{126}{27} = \frac{14}{3}$$

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27. $\int_{\bullet}^{\infty}\frac{\ln x\,dx}{2x+2x\ln^4 x}$

Let $y = \ln^2 x$

$dy = \frac{2\ln x}{x}\,dx$

$\int_{\bullet}^{\infty}\frac{\ln x}{x}\left(\frac{dx}{1+\ln^4 x}\right)$

$\int_{\bullet}^{\infty}\left(\frac{1}{4}\frac{dy}{1+y^2}\right)$

$\frac{1}{4}\int_{\bullet}^{\infty}\frac{dy}{1+y^2}$

$\frac{1}{4}\left(\tan^{-1} y\Big]_{\bullet}^{\infty}\right) = \frac{1}{4}\left(\tan^{-1}(\ln^2 x)\right)\Big]_1^{\infty}$

$\frac{1}{4}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{8}$

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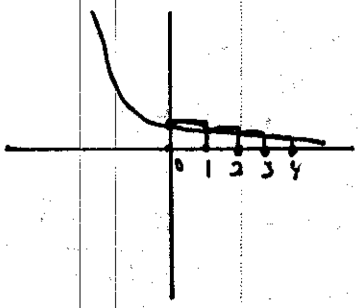
28.



$h = \frac{b-a}{n}$

$A \approx \frac{4-0}{4}\left(f(0)+f(1)+f(2)+f(3)\right)$

$A \approx 1+e^{-1}+e^{-2}+e^{-3}$

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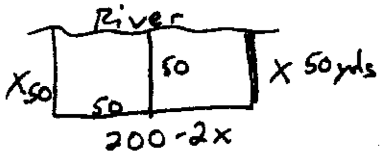
29.



$A = 200x - 2x^2$

$200 - 4x = 0$

$x = 50$

50yds x 50yds = 150ft by 150ft

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30. $e^{xy} = x$

$xy = \ln x$

$y = \frac{\ln x}{x}$

$y' = \frac{1 - \ln x}{x^2}$

$y' = \frac{1 - \ln 2}{4}$

(E)

Team Round

1. $y = -2x^3 + 6x^2 + 3x - 5 \quad y(-1) = 0$

$y' = -6x^2 + 12x + 3$

$y'(-1) = -6 - 12 + 3 = -15$

$y = \frac{1}{15}x + b$

$0 = -\frac{1}{15} + b$

$b = \frac{1}{15}$

$y = \frac{1}{15}x + \frac{1}{15}$

$\boxed{x - 15y + 1 = 0}$

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2. $\int x^3 e^{2x}\,dx$

Tic tac toe!

$x^3 \quad + \quad e^{2x}$

$3x^2 \quad - \quad \frac{1}{2}e^{2x}$

$6x \quad + \quad \frac{1}{4}e^{2x}$

$6 \quad - \quad \frac{1}{8}e^{2x}$

$0 \quad \quad \frac{1}{16}e^{2x}$

$\boxed{\frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C}$

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3. $MVT = \frac{f(b) - f(a)}{b - a} = f'(c)$

$\frac{f(4) - f(2)}{4 - 2} = 2c - 5$

$1 = 2c - 5$

$c = 3$

Rolle's Thm $\frac{f(b) - f(a)}{b - a} = f'(c)$, $f(b) - f(a) = 0$

$0 = 2c - 5$

$c = \frac{5}{2}$

$\cancel{2}(\frac{5}{\cancel{2}})(3) = \boxed{15}$

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4. $y = x^4 - 6x^2 - 8x + 80$

$y' = 4x^3 - 12x - 8 = 0$

$y' = 4(x+1)(x+1)(x-2)$

$\ominus \;|\; \ominus \;|\; \oplus$

$-1 \quad 2$

$y'' = 12x^2 - 12$

$\oplus \;|\; \ominus \;|\; \oplus$

$-1 \quad 1$

$\boxed{(-\infty, -1) \cup (1, 2)}$

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5. $\lim_{x \to 2} (3x - 5)^{\frac{1}{x^2 - 4}}$

$y = \lim_{x \to 2} (3x - 5)^{\frac{1}{x^2 - 4}}$

$\ln y = \lim_{x \to 2} \frac{\ln(3x - 5)}{x^2 - 4}$ L'Hopitals

$\ln y = \lim_{x \to 2} \frac{\frac{3}{3x - 5}}{2x}$

$\ln y = \frac{3}{4}$

$\boxed{y = e^{\frac{3}{4}}}$

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6. $f(x) = \sin^3 x$

$g'(x) = \frac{1}{f'(g(x))}$

$g'(\frac{3\sqrt{3}}{8}) = \frac{1}{f'(g(\frac{3\sqrt{3}}{8}))}$

$f(\frac{\pi}{3}) = \frac{3\sqrt{3}}{8}$

$\frac{\pi}{3} = f^{-1}(\frac{3\sqrt{3}}{8})$

$\frac{\pi}{3} = g(\frac{3\sqrt{3}}{8})$

$g'(\frac{3\sqrt{3}}{8}) = \frac{1}{f'(\frac{\pi}{3})}$

$f'(x) = 3\sin^2 x \cos x$

$f'(\frac{\pi}{3}) = 3(\frac{3}{4})(\frac{1}{2}) = \frac{9}{8}$

$g'(\frac{3\sqrt{3}}{8}) = \frac{1}{\frac{9}{8}}$

$\boxed{g'(\frac{3\sqrt{3}}{8}) = \frac{8}{9}}$