2005 Palm Harbor February Invitational
Calculus Answer Key

<u>Individual</u>

1. C
2. D
3. D
4. A
5. C
6. B
7. A
8. B
9. E
10. A
11. C
12. D
13. D
14. B
15. C
16. A
17. B
18. D
19. A
20. C
21. A
22. D
23. A
24. C
25. E
26. B
27. B
28. A
29. C
30. C

<u>Team</u>

1. 6
2. $2\sqrt{2}$
3. $\frac{109}{60}$
4. $\frac{\sqrt{2}}{4}$
5. B, C, D
6. 1162
7. 4.31
8. 135
9. $\frac{\sqrt{2}}{2}$
10. C, D, B, A
11. 86
12. 0.412
13. 12.5
14. 0.0495

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Calculus Team Solutions

1) The area of 1 hump will be the same. $\int_0^{\frac{\pi}{A}} \sin(Ax)dx = \frac{2}{A}$

Since $\sin(Ax)$ intersects the x-axis whenever $x = \frac{\pi n}{A}$, the number of humps is

$\pi A = \frac{\pi n}{A} \Rightarrow n = A^2$. Total area = number of humps × area of 1 hump = $\frac{2}{A} \cdot A^2 = 2A = 12$

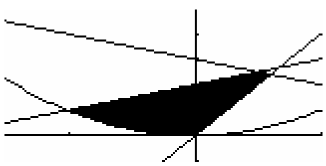
$A = 6$ **Answer: 6**

2) $A = \int_0^C \frac{dx}{x^2 + C^2} = \frac{1}{C}\arctan\left(\frac{x}{C}\right)\Big|_0^C = \frac{\pi}{4C}$ $\quad B = Ax^{A-1}\Big|_{x=2} = A(2)^{A-1}$

$y'' = -B^2 \sin(Bx) = 0$, $x = 0, \pm\frac{\pi}{B}, \pm\frac{2\pi}{B}, \ldots$ $\quad C = \frac{\pi}{B}$

$A = \frac{\pi}{4C} = \frac{B}{4} = \frac{A(2)^{A-1}}{4} \Rightarrow 2^{A-1} = 4$ $\quad A = 3$, $B = 12$, $C = \frac{\pi}{12}$

$\int_0^{\pi/12} 12\cos(3x)dx = 2\sqrt{2}$ **Answer:** $2\sqrt{2}$

3) 

The area of the shaded region can be defined by the integral:

$A = \int_{-1}^{0}\left[(x+2) - x^2\right]dx + \int_{0}^{0.5}\left[(x+2) - 4x\right]dx + \int_{0.5}^{0.6}\left[(-x+3) - 4x\right]dx = \frac{7}{6} + \frac{5}{8} + \frac{1}{40} = \frac{109}{60}$

Answer: $\frac{109}{60}$

4) Use Shell Method: $2\pi\int_0^a x \cdot x^{2/3}dx = \frac{3\pi}{4}x^{8/3}\Big|_0^a = \frac{3\pi}{4}a^{8/3} = \frac{3\pi}{64} \Rightarrow a = \frac{\sqrt{2}}{4}$

Answer: $\frac{\sqrt{2}}{4}$

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5) I. $\lim_{x\to 0^+} f(x)=\infty$, $\lim_{x\to 0^-} f(x)=0$ FALSE

II. $f'(x)=-\frac{e^{1/x}}{x^2}$, $f''(x)=\frac{e^{1/x}+2xe^{1/x}}{x^4}=0 \quad x=-1/2$ TRUE

III. $\lim_{x\to\infty}\frac{-e^{1/x}}{x^2}=0$ TRUE

IV. $f'(x)$ does not equal 0 at any point x, but it is undefined at $x = 0$. However, since 0 is not a part of the original domain, it cannot be considered as a critical point. TRUE

V. $f'(x)$ is always decreasing, but it is concave down for all values of $x < -0.5$. FALSE

B, C, and D are true **Answer: B, C, D**

6) $y=ax^3+bx^2+cx+d$ $\quad y'=3ax^2+2bx+c$ $\quad y''=6ax+2b$

$(2,4)$: $4=8a+4b+2c+d$, $0=12a+2b$

$(1,-2)$: $-2=a+b+c+d$, $0=3a+2b+c$

Solving the system of equations gives: $a=-3$, $b=18$, $c=-27$, $d=10$

$a^2+b^2+c^2+d^2=1162$ **Answer: 1162**

7) A: There are 6 local minima including the one at $x=0$.

B: There are 11 inflection points.

C: There are 5 local maxima.

Midpoint rectangle approximation of $g(x)$ on the domain $[6,11]$ with 5 rectangles

Area = $(1)[g(6.5)+g(7.5)+g(8.5)+g(9.5)+g(10.5)]=4.31$

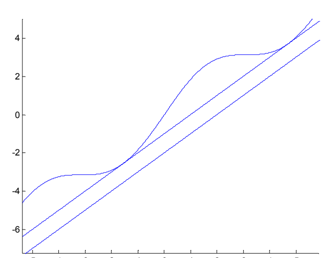
Answer: 4.31

8) $$P(x)=\begin{cases}1.4x & x\le 100\\ 140+[1.4-0.02(x-100)](x-100) & x>100\end{cases}$$

$$P'(x)=\begin{cases}1.4 & x\le 100\\ 5.4-0.04x & x>100\end{cases},\ 5.4-0.04x=0\Rightarrow x=135$$

Answer: 135

9) The line tangent to the graph $y=\sin(x)+x$ at the points closest to the line $y=x-2$ is the line $y=x-1$. These two lines are parallel at a distance of $\frac{\sqrt{2}}{2}$ between each other.



Answer: $\frac{\sqrt{2}}{2}$

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10) $A = \int_0^2 \frac{dx}{9+x^2} = \arctan\left(\frac{x}{3}\right)\Big|_0^2 = \arctan\left(\frac{2}{3}\right) \approx .196$

$B = \int_0^1 (x^3 + 2x^2 - 7x + 4)dx = \frac{x^4}{4} + 2\frac{x^3}{3} - 7\frac{x^2}{2} + 4x\Big|_0^1 = \frac{17}{12} \approx 1.417$

$C = \int_1^2 \left(\frac{1}{x} + 2^x\right)dx = \ln x + \frac{2^x}{\ln 2}\Big|_1^2 = \ln 2 + \frac{4}{\ln 2} - \frac{2}{\ln 2} \approx 3.579$

$D = \int_0^3 \frac{8dx}{15+2x-x^2} = -8\int_0^3 \frac{dx}{x^2-2x-15} = -8\int_0^3 \left(\frac{1/8}{x-5} - \frac{1/8}{x+3}\right)dx = -\left[\ln|x-5| - \ln|x+3|\right]_0^3 = \ln 5 \approx 1.609$

Highest to lowest: **C, D, B, A**

11) $y = x^3 + 2x^2 + 7x + 5$ $\qquad y\big|_{x=-1} = -1$

$\frac{dy}{dx} = 3x^2 + 4x + 7$ $\qquad \frac{dy}{dx}\Big|_{x=-1} = 6$

Tangent line: $y + 1 = 6(x+1)$ $\qquad y = 6x + 5$

$y = x^3 - 3x^2 - 4x + 7$ $\qquad y\big|_{x=1} = 1$

$\frac{dy}{dx} = 3x^2 - 6x - 4$ $\qquad \frac{dy}{dx}\Big|_{x=1} = -7$

Tangent line: $y - 1 = -7(x-1)$ $\qquad y = -7x + 8$

$6x + 5 = -7x + 8 \quad \Rightarrow \quad x = \frac{3}{13} \quad \Rightarrow \quad y = \frac{83}{13}$

$13(x+y) = 86$

12) After 5 seconds: width = 2.5 cm, length = 2.8 cm, height = 2.1 cm

$A = 2wl + 2wh + 2lh$

$\frac{dA}{dt} = 2\left(w\frac{dl}{dt} + l\frac{dw}{dt} + w\frac{dh}{dt} + h\frac{dw}{dt} + l\frac{dh}{dt} + h\frac{dl}{dt}\right) =$

$2(2.5(0.01 - 0.02) + 2.1(0.05 - 0.02) + 2.8(0.05 + 0.01)) = 0.412$

Answer: 0.412

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13) At $x = 1$, the tangent line is $y = 2x - 1$. At $x = 2$, the tangent line is $y = 4x - 4$

The angle between the two graphs is $\tan\phi = \dfrac{m_2 - m_1}{1 + m_1 m_2} = \dfrac{2}{9}$ $\phi = 12.5^\circ$

Answer: 12.5

14) Law of cosines: $y^2 = x^2 + x^2 - 2x^2 \cos\theta$ $\qquad 2y\dfrac{dy}{dt} = 2x^2 \sin\theta \dfrac{d\theta}{dt}$

$$\frac{dy}{dt} = \frac{x^2 \sin\theta}{2x^2 - 2x^2\cos\theta}\frac{d\theta}{dt} = \frac{10^2 \sin(20^\circ)}{2(10^2) - 2(10^2)\cos(20^\circ)}\frac{1^\circ \pi}{180^\circ} = 0.0495$$

Answer: 0.0495