2003 Palm Harbor Invitational Calculus Team Questions

1. The volume of an ellipsoid is given by $V = \frac{4}{3}\pi abc$ where a, b, and c are the lengths of the semi-axes. Initially, $a = 18$ meters, $b = 12$ meters, and $c = 15$ meters. a increases at a rate 2 meters per second, b increases at a rate of 4 meters per second, and c increases at a rate of 3 meters per second. Find the exact rate of change of the volume of the ellipsoid when the ellipsoid is a sphere (in cubic meters per second).

2. Let $f(x) = \sum_{i=1}^{\infty} \frac{d^i\left[2^x\right]}{dx^i}$. Find the exact value of $f(0)$.

3. Let $g(x) = f\left(x^2\right)$. If $f(2) = 1$, $f'(2) = 8$, and $f''(2) = 2$, find $g''\left(\sqrt{2}\right)$.

4. Find all non-trivial solutions to the differential equation $\frac{dy}{dx} = y^2(2x-1)$.

5. Below are seven statements concerning continuity. If the statement is true, its value appears next to it in parentheses. If the statement is false, its value is 0. Find the sum of all the statements' values.

(-3) $f(x)$ is continuous at $x = a$ provided that $\lim_{x\to a} f(x) = l$, $f(a)$ exists, and $f(a) = l$

(12) Continuity implies differentiability

(-23) Differentiability implies continuity

(9) $f(x)$ is continuous on the interval $[a,b]$ if and only if $\lim_{x\to c} f(x) = f(c)$ for $a < x < b$, $\lim_{x\to a^+} f(x) = f(a)$, and $\lim_{x\to b^-} f(x) = f(b)$

(31) If $f(x)$ and $g(x)$ are continuous at $x = a$, then $f(x)g(x)$ is also continuous at $x = a$

(-43) If $f(x)$ and $g(x)$ are continuous at $x = a$, then $\frac{f(x)}{g(x)}$ is also continuous at $x = a$

(15) If $f(x)$ is continuous on the interval $[a,b]$, $f(a) = A$, and $f(b) = B$, then corresponding to any number C between A and B there exists exactly one number c in $[a,b]$ such that $f(c) = C$

6. If a bar chart is made where the top of the bars were described by $f(x)=\left[x^2\right]$, where $[x]$ is the greatest integer less than or equal to x and the bottom of the bars lie on the x-axis, what is the total area of the bars from $x=-1$ to $x=3$? Round your answer to the nearest thousandth.

7. Let $f(x)=-x^2\cos x$ and $g(x)=-x^2\sin x$.

$A=f'(\pi)$

$B=g''(\pi)$

$C=$ The number of times $f(x)=g(x)$ on $[-\pi,\pi]$

$D=$ The number of times the graphs of $f(x)$ and $g(x)$ are tangent to each other on $[-\pi,\pi]$

Evaluate $\int_A^B\left[\sin^C x\cos x+D\right]dx$.

8. The area bounded by the x-axis and $y=\sin x$ on the interval $[-p,p]$ is equal to 29. Find p.

9. Evaluate: $\int_0^\infty xe^{-ax}\,dx$ where a is a positive number.

10. $xf(x)-xf(c)=xf(x+c)-cf(x+c)$ and $f(4)=\pi$ where c is a constant. Find the exact value of $f'(2)$.

11. The graphs of $f(x)=\cos x$ and $g(x)=ax^2+b$, where a and b are constants, are tangent to each other when $x=2$. Find $a+b$ rounded to the nearest thousandth.

12. Let A, B, and C be real numbers given by $A=e^{\int_1^{C^2}\frac{1}{x}dx}$, $B=\int_0^1 x^A\,dx$, and $C=\int_{2B}^\infty\frac{1}{x^2}dx$.

Find $\int_A^C x^Be^{-x}\,dx$.

13. Given that $\sqrt{1.96}=1.4$, find the exact value of the approximation of $\sqrt{2}$ using differentials.

14. Find the exact slope of the line tangent to the graph of $xy^2 + yx^2 + 6x + 8y = 28$ at the point $(1,2)$.

15. Find the exact value of: $\int_0^{\frac{1}{2}} \sqrt{1-4x^2}\, dx$