2004 Palm Harbor Invitational Calculus Individual Solutions

1) $g(x) = x^3 + 5x^2 - 8x + 2$

$g'(x) = 3x^2 + 10x - 8 = 0 \quad x = -4, \frac{2}{3}$

$g(-5) = 42$; $g(-4) = 50$; $g\left(\frac{2}{3}\right) = -\frac{22}{27}$; $g(4) = 114$

114 is the maximum value **C**

2) f has a cusp at $x = 0$
g has a vertical asymptote at $x = 0$
h has a vertical asymptote at $x = 0$
j has a vertical tangent at $x = 0$
f is the only one with a non-differentiable cusp **A**

3) Using L'Hopital's rule:

$$\lim_{x\to 0} \frac{\frac{d}{dx}\left[\int_0^x (t^2 - t + 2)dt\right]}{\frac{d}{dx}[x]} \quad \lim_{x\to 0}(x^2 - x + 2) = 2 \quad \textbf{C}$$

4) The point on $h(x)$ is $\left(x, 2^{1-x^2}\right)$ Area: $A(x) = x \cdot 2^{1-x^2}$

$A'(x) = x \cdot \left(-2x \cdot \ln 2 \cdot 2^{1-x^2}\right) + 2^{1-x^2} = 0$

$1 - 2x^2 \ln 2 = 0 \quad x = \sqrt{\frac{1}{\ln 4}} = a \quad 2^{1-\frac{1}{\ln 4}} = 2e^{-1/2} = b$

$a + b \approx 2.06$ **C**

5) $m = \frac{21-3}{3-1} = \frac{18}{2} = 9 \quad 4x^3 - 24x^2 + 44x - 15 = 9$

$x = 1,2,3$ Since endpoints don't count, sum is 2 **A**

6) $\int_0^{\frac{\pi}{4}} \sec^2 x \tan x dx$ Let $u = \tan x, du = \sec^2 x dx$

$\frac{1}{2}(\tan x)^2 \Big]_0^{\frac{\pi}{4}} = \frac{1}{2}$ **B**

7) $\lim_{x\to 0} \frac{\sin 2x - 2\sin x}{x^2} \quad \lim_{x\to 0} \frac{2\cos x \sin x - 2\sin x}{x^2}$

$\lim_{x\to 0} \frac{2\sin x}{x} \cdot \frac{\cos x - 1}{x} \quad 2 \cdot 0 = 0$

A

8) $y = ax^3 + bx^2 - 2x + 1 \quad y(1) = a + b - 1 = 1$
$y' = 3ax^2 + 2bx - 2 \quad a + b = 2$
$y'' = 6ax + 2b \quad a - 3a = 2$
$y''(1) = 6a + 2b = 0 \quad a = -1 \quad b = 3$
$b = -3a \quad a^2 + b^2 = 10$ **C**

9) $x^3 + 2x + 5 = 2 \quad x = -1$

$g'(2) = \frac{1}{f'(-1)} = \frac{1}{3(-1)^2 + 2} = \frac{1}{5}$ **D**

10) The graph is a quarter of a circle. When revolved, it forms half the surface area of a sphere. Since $r = \frac{1}{2}$ and surface area of entire sphere $= 4\pi r^2$, surface area of the region created $= \frac{1}{2}\left(4\pi\left(\frac{1}{2}\right)^2\right) = \frac{\pi}{2}$ **B**

11) I. False – the graph of f does not change concavity, therefore there is not an inflection point.
II. True – same reason for I.
III. False – for example: $a = 1, b = 20$
$f(1) = 21 \quad f'(1) = 24 \quad f''(1) = 12$
IV. True – $f'(x) = ax^3 + b$, the only inflection point will be at $x = 0$ II. & IV. are true. **B**

12) $x = 1 \quad y^2 - 3y - 4 = 0 \quad y = -1, 4 \quad P(x,y) = (1,-1)$

$\frac{dy}{dx} = \frac{6xy - y^2}{2xy - 3x^2} = \frac{7}{5} = m_{\tan} \quad m_{norm} = -\frac{5}{7}$

$-7(y+1) = 5(x-1) \Rightarrow 5x + 7y = -2$ **A**

13) $d = \sqrt{(x-0)^2 + (x^2 - 8 - 5)^2} = \sqrt{x^4 - 25x^2 + 169}$

$d' = \frac{2x^3 - 25x}{\sqrt{x^4 - 25x^2 + 169}} = 0 \quad x = 0, \pm\frac{5\sqrt{2}}{2}$

$\sqrt{\left(\frac{5\sqrt{2}}{2}\right)^4 - 25\left(\frac{5\sqrt{2}}{2}\right)^2 + 169} = \frac{\sqrt{51}}{2}$ **D**

14) $\ln(T - T_c) = kt + c \quad T_c = 27°\text{C} \quad \ln(95 - 27) = \ln(68) = c$

$\ln(65 - 27) = k(2) + \ln(68) \quad k = \frac{1}{2}\ln\frac{19}{34}$

$\ln(40 - 27) = \left(\frac{1}{2}\ln\frac{19}{34}\right)t + \ln(68) \quad t \approx 5.7$ min **E**

15) $\sum_{i=1}^{n}\left(\frac{i^6}{n^7}\right) = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^6\left(\frac{1}{n}\right)$

From the properties of integrals

$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^6\left(\frac{1}{n}\right) = \int_0^1 x^6 dx = \frac{1}{7}$ **C**

16) $\ln(e-1) - \ln e - \frac{1}{e} \approx 0.632$ **D**

17) A) has cusp at x=0; so non-differentiable on interval
B) has removable discontinuity at so non-differentiable on interval
C) at $x = -4$; $y = 1$, at $x = 4$; $y = 4$, endpoints not equal
D) continuous on, $[-4,4]$, differentiable on $(-4,4)$ and endpoints both have y-value of 0 **D**

2004 Palm Harbor Invitational Calculus Individual Solutions

18) average value: $\frac{1}{4-2}\int_2^4 \ln x\,dx$

Using integration by parts:

$\left.\frac{x}{2}\ln x-\frac{x}{2}\right]_2^4=3\ln 2-1=\ln 8-1$ **C**

19) Using the Shell Method:

$2\pi\int_0^1 x\left[(x+1)-(x^2+1)\right]dx=2\pi\int_0^1 (x^2-x^3)dx=$

$2\pi\left(\left.\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1\right)=\frac{\pi}{6}$ **A**

20) $\lim_{x\to\infty}\frac{4x^2+2x+3x^3}{5x^3+4x^2+x+1}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}}=\lim_{x\to\infty}\frac{\frac{4}{x}+\frac{2}{x^2}+3}{5+\frac{4}{x}+\frac{1}{x^2}+\frac{1}{x^3}}=\frac{3}{5}$

B

21)

nth derivative	expression
1	$\pi(-\sin(\pi x)+i\cos(\pi x))$
2	$\pi^2(-\cos(\pi x)-i\sin(\pi x))$
3	$\pi^3(\sin(\pi x)-i\cos(\pi x))$
4	$\pi^4(\cos(\pi x)+i\sin(\pi x))$
2004	$\pi^{2004}(\cos(\pi x)+i\sin(\pi x))$

$\cos(\pi x)+i\sin(\pi x)=e^{i\pi x}$ $\quad\pi^{2004}e^{i\pi x}$ **C**

22) A) False – f' has a local minimum between $x=2$ and $x=3$

B) False – f does not have to cross the x-axis

C) True – $f''<0$ on the interval (1,2)

D) False – f'' has a local minimum on the interval (1,2)

Answer: **C**

23) When perpendicular to the y-axis, then a length of x is used.

$$x=\sqrt{25\left(1-\frac{y^2}{9}\right)}=\frac{5}{3}\sqrt{9-y^2}$$

Area of eq. triangle: $\frac{s^2\sqrt{3}}{4}=\frac{(2x)^2\sqrt{3}}{4}=x^2\sqrt{3}$

$\frac{25\sqrt{3}}{9}\int_{-3}^{3}(9-y^2)dy=\frac{25\sqrt{3}}{9}(36)=100\sqrt{3}$ **D**

24) $x^2dy-y^2dx=4dx$ $\quad\frac{dx}{x^2}=\frac{dy}{y^2+4}$

Integrating both sides: $\quad-\frac{1}{x}=\frac{1}{2}\arctan\left(\frac{y}{2}\right)+C$

$-\frac{1}{1}=\frac{1}{2}\arctan\left(\frac{0}{2}\right)+C$ $\quad C=-1$

$\frac{1}{2}\arctan\left(\frac{y}{2}\right)=1-\frac{1}{x}$ $\quad\arctan\left(\frac{y}{2}\right)=2-\frac{2}{x}$ **A**

25) x = distance from robot to lightpole

y = distance from tip of shadow to lightpole

$\frac{dx}{dt}=4$

Using triangle proportions:

$\frac{y}{20}=\frac{y-x}{8}$ $\quad y=\frac{5x}{3}$

$\frac{dy}{dt}=\frac{5}{3}\frac{dx}{dt}=\frac{5(4)}{3}=\frac{20}{3}$ **E**

26) $\sum_{n=1}^{\infty}\frac{(x-1)^n(-1-x)^n}{n!}=\sum_{n=1}^{\infty}\frac{(1-x^2)^n}{n!}=$

$\left(\sum_{n=0}^{\infty}\frac{(1-x^2)^n}{n!}\right)-1=e^{1-x^2}-1$ **D**

27) $\int\frac{2dx}{x\sqrt{x^4-1}}=\int\frac{2xdx}{x^2\sqrt{x^4-1}}$ $\quad u=x^2$ $\quad du=2xdx$

$=\sec^{-1}(x^2)+C$ **D**

28) $y=\sin x-x^2$ $\quad y'=\cos x-2x$

$x_{n+1}=x_n-\frac{\sin x_n-x_n^2}{\cos x_n-2x_n}$

After two iterations $x_2=1.329$ **B**

29) $a=x(13.2-x)^2$ $\quad a'=-2x(13.2-x)+(13.2-x)^2=0$

$2x=13.2-x$ $\quad x=4.4$

Since 4.4 is being taken off from both sides of the box, the height is 2.2: $\quad a=2.2(8.8)^2=170.368$

$\ln 170.368=5.138...$ **A**

30) $V=\frac{1}{3}\pi r^2h$ $\quad V=6\pi$ $\quad h=2$ $\quad\frac{dr}{dt}=3$

$6\pi=\frac{1}{3}\pi r^2(2)$ $\quad r=3$

$\frac{dV}{dt}=\frac{1}{3}\pi r^2\frac{dh}{dt}+\frac{1}{3}\pi 2rh\frac{dr}{dt}=0$

$(3^2)\frac{dh}{dt}=-2(3)(2)(3)$ $\quad\frac{dh}{dt}=-4$ **B**