2003 Palm Harbor Invitational Calculus Answers

Individual

1. A
2. C
3. B
4. C
5. C
6. B
7. B
8. B
9. D
10. A
11. B
12. D
13. A
14. D
15. E
16. D
17. B
18. D
19. C
20. A
21. D
22. B
23. B
24. E
25. A
26. A
27. B
28. B
29. C
30. C

Team

1. $.6912\pi$
2. $\dfrac{\ln 2}{1-\ln 2}$
3. 32
4. $y=\dfrac{-1}{x^2-x+C}$
5. 14
6. 7.694
7. 2π
8. $\dfrac{22\pi}{3}$
9. $\dfrac{1}{a^2}$
10. $\dfrac{\pi}{2}$
11. .266
12. 0
13. $\dfrac{99}{70}$ or $1\dfrac{29}{70}$
14. $-\dfrac{14}{13}$
15. $\dfrac{\pi}{8}$

2003 Palm Harbor Invitational Calculus Individual Solutions

1. $f(x)$ and $g(x)$ intersect when $x=-3$ and $x=-2$. The area of the region of intersection is given by $\int_{-3}^{-2}\left[x-7-\left(x^2+6x-1\right)\right]dx=\frac{1}{6}$. **A**

2. $\lim_{x\to 2}\frac{x^3+x^2+ax-2}{x^2-x-2}=\frac{8+4+2a-2}{4-2-2}=\frac{10+2a}{0}$. In order for the limit to equal a real number, $10+2a$ must equal zero. Thus $a=-5$.
$\lim_{x\to 2}\frac{x^3+x^2-5x-2}{x^2-x-2}=\lim_{x\to 2}\frac{3x^2+2x-5}{2x-1}=\frac{12+4-5}{4-1}=\frac{11}{3}$. $\frac{11}{3}-5=-\frac{4}{3}$ **C**

3. $f'(x)=4x^3-12x^2-40x+96$ $x^3-3x^2-10x+24=0$ $x=-3$, $x=2$, and $x=4$. Since we are concerned with the interval $[-2,5]$, we disregard $x=-3$. $f(2)=208$, $f(4)=176$, $f(-2)=-112$, and $f(5)=217$. Thus the minimum is -112. **B**

4. $\int_a^b (a-x)(x-b)dx=\int_a^b\left(-x^2+(a+b)x-ab\right)dx=-\frac{x^3}{3}+\frac{(a+b)x^2}{2}-abx\Big]_a^b=$

$-\frac{b^3}{3}+\frac{ab^2}{2}+\frac{b^3}{2}-ab^2+\frac{a^3}{3}-\frac{a^3}{2}-\frac{a^2b}{2}+a^2b=\frac{b^3}{6}-\frac{ab^2}{2}+\frac{a^2b}{2}-\frac{a^3}{6}=12$
$b^3-3ab^2+3a^2b-a^3=(b-a)^3=72$ $\quad b-a=\sqrt[3]{72}=2\sqrt[3]{9}$. **C**

5. $f(x-\Delta x)\approx f(x)-f'(x)\Delta x$. $45°=\frac{\pi}{4}$ radians and $40°=\frac{2\pi}{9}$ radians.
$\sin 40°=\sin\frac{2\pi}{9}\approx\sin\frac{\pi}{4}-\cos\frac{\pi}{4}\cdot\left(\frac{\pi}{4}-\frac{2\pi}{9}\right)\approx .645$ **C**

6. $f'(x)=\frac{(2x+3)(x+1)-\left(x^2+3x-2\right)}{(x+1)^2}=\frac{x^2+2x+5}{(x+1)^2}$

$f''(x)=\frac{(2x+2)(x+1)^2-2(x+1)\left(x^2+2x+5\right)}{(x+1)^2}=\frac{-8}{x+1}$

$f''(x)$ is undefined at $x=-1$. $f(x)$ is concave up when $f''(x)$ is positive.
$f''(x)$ is positive on $(-\infty,-1)$. **B**

7. The eccentricity of an ellipse is equal to the ratio of its focal distance (c) to the length of it semi-major axis (a). As the foci move closer to the center, the ellipse will approach a circle. $\lim_{c \to 0} \frac{c}{a} = 0$. **B**

8. $\frac{2}{n^2-1} = \frac{A}{n-1} + \frac{B}{n+1}$ $\quad 2 = A(n+1) + B(n-1)$ $\quad A = 1$ and $B = -1$

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty} \left[\frac{1}{n-1} - \frac{1}{n+1}\right] = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \ldots = \frac{3}{2}$$ **B**

9. The unshaded area can be divided into eight congruent parts, each with an area of $4-\pi$. The area of the unshaded region is therefore $8(4-\pi) = 32 - 8\pi$. The area of the shaded is therefore $16 - (32 - 8\pi) = 8\pi - 16$. **D**

NOTE: The area can also be represented by the integral

$$4\int_0^2 \left[\sqrt{4-(x-2)^2} + \sqrt{4-x^2} - 2\right] dx = 8\pi - 16$$

10. $\int_0^7 \frac{x}{\sqrt{x^2+1}} dx = \frac{1}{2}\int_0^7 \frac{2x}{\sqrt{x^2+1}} dx = \sqrt{x^2+1}\Big]_0^7 = \sqrt{50} - \sqrt{1} = 5\sqrt{2} - 1$ $\quad 5 + 2 - 1 = 6$ **A**

11. The length of the diagonal is $d = s\sqrt{3}$, so $\frac{dd}{dt} = \sqrt{3}\frac{ds}{dt}$. $\frac{dd}{dt} = 5$ $\quad \frac{ds}{dt} = \frac{5\sqrt{3}}{3}$. $V = s^3$

$\frac{dV}{dt} = 3s^2 \frac{ds}{dt} = 3 \cdot 4 \cdot \frac{5\sqrt{3}}{3} = 20\sqrt{3}$ **B**

12. The area of each cross-section is $A(x) = \frac{\pi\left(\sqrt{9-x^2}\right)^2}{2}$. The volume of the solid is therefore given by $\int A(x)dx = \frac{\pi}{2}\int_{-3}^{3}(9-x^2)dx = 18\pi$. **D**

13. $E_d = \frac{2500 - .01x - \frac{1849}{x}}{2500 - .02x} = 1$ $\quad 2500 - .01x - \frac{1849}{x} = 2500 - .02x$ $\quad .01x = \frac{1849}{x}$

$x^2 = 184900$ $\quad x = 430$ **A**

14. The roots of $x^3 - 2x^2 - 5x + 6$ are $x = -2$, $x = 1$, and $x = 3$. The fourth quadrant area is given by $-\int_1^3 (x^3 - 2x^2 - 5x + 6)dx = \frac{16}{3}$ **D**

15. The Maclaurin series expansion of $\sin x$ is equal to $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}$. The coefficient of the x^n term is 0 when n is even. Therefore the coefficient of the x^{64} term is 0. **E**

16. **D**

17. $\int_1^2 x^{2x}(1+\ln x)dx=\int_1^2 x^x\cdot x^x(1+\ln x)dx$ Let $u=x^x$ and $du=x^x(\ln x+1)dx$. Thus

$\int_1^2 x^{2x}(1+\ln x)dx=\int_1^4 u\,du=\left.\frac{u^2}{2}\right]_1^4=8-\frac{1}{2}=\frac{15}{2}$ **B**

18. $V=x(10-2x)(12-2x)=4x^3-44x^2+120$ $12x^2-88x+120=0$ $x=\frac{88\pm\sqrt{1984}}{24}$.

x must be less than 5, thus $x=\frac{88-\sqrt{1984}}{24}$. $V\approx 96.77$ **D**

19. $\frac{d[A]}{dt}=-k[A]$ $\frac{d[A]}{[A]}=-kdt$ $\ln[A]=-kt+C$. Let the initial concentration be $[A]_0$. $C=\ln[A]_0$. The concentration at the half-life is equal to $t_{1/2}=\frac{[A]_0}{2}$. Therefore,

$\ln\left(\frac{[A]_0}{2}\right)=-kt+\ln[A]_0$. Simplifying yields

$t_{1/2}=\frac{\ln 2}{1.32\times10^{-5}s^{-1}}=52511.15\text{ seconds}=14.59\text{ hours}$ **C**

20. $\frac{x^3-7x-6}{x+1}=x^2-x-6=14$ $a^2-a-20=0$. $a=5$ and -4

$5^2+(-4)^2=25+16=41$. **A**

21. $f^1(x)=\frac{2}{x}$ $f^2(x)=\frac{-2}{x^2}$ $f^3(x)=\frac{4}{x^3}$ $f^4(x)=\frac{-12}{x^4}$

$f^n(x)=(-1)^{n+1}2(n-1)!x^{-n}$ **D**

22. $A=\frac{1}{2}(10-x)(2x-4)\sin 45°=\frac{\sqrt{2}}{4}(-2x^2+24x-40)$ $\frac{\sqrt{2}}{4}(24-4x)=0$

$x=6$ $A=8\sqrt{2}$ **B**

23. $f(c)=\frac{1}{4-2}\int_2^4 (x^3-8)\,dx=22 \quad c^3-8=22 \quad c=\sqrt[3]{30}$ **B**

24. $f'(x)=12x^2-12x+3 \quad f''(x)=24x-12 \quad 12a^2-12a+3=24a-12$

$12a^2-36a+15=0 \quad a=\frac{5}{2}$ and $a=\frac{1}{2}$. $\frac{5}{2}+\frac{1}{2}=3$ **E**

25.

$\int_0^1 \frac{4}{1+x^2}\,dx \approx \frac{1}{10}\left(\frac{4}{1+0^2}+2\cdot\frac{4}{1+.2^2}+2\cdot\frac{4}{1+.4^2}+2\cdot\frac{4}{1+.6^2}+2\cdot\frac{4}{1+.8^2}+\frac{4}{1+1^2}\right)\approx 3.135$ **A**

26. **A**

27. $\frac{dT}{dt}=k(T-20) \quad \frac{dT}{T-20}=k\,dt \quad \ln|T-20|=kt+C$

$t=0 \Rightarrow T=90 \quad C=\ln 70$

$t=5 \Rightarrow T=60 \quad \ln 40=5k+\ln 70 \quad k=\frac{1}{5}\ln\frac{4}{7}$

$\ln|T-20|=\left(\frac{1}{5}\ln\frac{4}{7}\right)t+\ln 70 \quad \ln|T-20|=2\ln\frac{4}{7}+\ln 70=\ln\frac{16\cdot 70}{49}=\ln\frac{160}{7}$

$T-20=\frac{160}{7} \quad T=\frac{300}{7}\approx 42.857$. Rounds to 43. **B**

28. **B**

29. Using integration by parts, $u=\arcsin x$, $du=\frac{1}{\sqrt{1-x^2}}dx$, $dv=dx$, and $v=x$. Thus

$\int \arcsin x\,dx = x\arcsin x-\int\frac{x}{\sqrt{1-x^2}}dx = x\arcsin x+\sqrt{1-x^2}+C$ **C**

30. $g(x)=2x^2-2x$, $g'(x)=4x-2$. $f'(2)=2g(2)g'(2)+g'(5)=2\cdot 4\cdot 6+18=66$ **C**