Calculus Solutions February 2002 Invitational

1. The limit gives the indeterminate form $\frac{0}{0}$. L'Hôpital's rule must be used. Taking the derivative of the numerator and the denominator yields $\lim_{x\to 2}\frac{2x+1}{3x^2-8x+1}=\frac{5}{-3}$. **A**

2. The total equals $\int_0^9\left[t^2-\sqrt{t}\right]dt=225$. **C**

3. Since $y=0$ is a horizontal asymptote, x approaches infinity as $y=\frac{1}{x^2}$ approaches the x-axis. As a result, the integral is improper, and the area is given by

$$\lim_{x\to\infty}\int_1^b\frac{1}{x^2}\,dx=\lim_{x\to\infty}\left[\frac{-1}{x}\right]_1^b=\lim_{x\to\infty}\left[\frac{-1}{b}-\frac{-1}{1}\right]=0+1=1.$$ **C**

4. Let $f(x)=\sqrt[5]{x}$, so $f'(x)=\frac{1}{5\sqrt[5]{x^4}}$. Since $\sqrt[5]{243}=3$, we let $\sqrt[5]{250}=\sqrt[5]{243+\Delta x}$ and $\Delta x=7$. The approximation is $f(x)+f'(x)\Delta x=3+\frac{7}{5\cdot 243^{\frac{4}{5}}}=3.0173$. **D**

5. $\int_5^a\frac{2x-5}{x^2-5x+8}dx=\ln\left|x^2-5x+8\right|_5^a=\ln\left|a^2-5a+8\right|-\ln 8=\ln\frac{\left|a^2-5a+8\right|}{8}=\ln\frac{1}{4}$

$\frac{\left|a^2-5a+8\right|}{8}=\frac{1}{4}$ so $\left|a^2-5a+8\right|=2$. $a^2-5a+6=0$ and $a^2-5a+10=0$. The real values of a are 3 and 2, so the sum is 5. **B**

6. $f'(2)=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$. Choosing $x=1.99999$, the limit equals 3.04. **A**

7. I. True

II. False, the marginal average cost is equal to $\left(\frac{C(x)}{x}\right)'$

III. True
IV. True
I, III, and IV are true. **E.**

8. $\frac{dy}{dx} = 1 + 3x^2 + y^2 + 3x^2y^2 = (1+3x^2)(1+y^2)$

$\int \frac{1}{1+y^2}\ dy = \int (1+3x^2)\ dx$

$\arctan y = x + x^3 + C$ **C**

$y = \tan(x^3 + x + C)$

9. Let $u = \sec x$, $du = \tan x \sec x\ dx$. $\int u^2\ du \to \frac{\sec^3 x}{3} + C$ **D**

10. I. True

II. False. $f(x)$ must be differentiable on (a,b)

III. True

I and III are true. **B**

11. $\int_{11}^{-4} [f(x)+4]\ dx = \int_{11}^{-4} f(x)\ dx + \int_{11}^{-4} 4\ dx = -(m+n) + 4x]_{11}^{-4} = -m - n - 60$.

$-1 - 1 - 60 = -62$. **A**

12. Since $f''(x) = 6x + 8$, $f'(x) = 3x^2 + 8x + C_1$ and $f(x) = x^3 + 4x^2 + C_1x + C_2$. $f(2) = 21$ and $f(1) = 14$, so $2C_1 + C_2 = -3$ and $C_1 + C_2 = 9$. Therefore $C_1 = -12$, $C_2 = 21$, and $f(x) = x^3 + 4x^2 - 12x + 21$. $f(3) = 27 + 36 - 36 + 21 = 48$ **D**

13. $f^n(x) = (-1)^{n+1} ne^{-x} + (-1)^n xe^{-x}$

$f^n(n) = (-1)^{n+1} ne^{-n} + (-1)^n ne^{-n} = (-1)^n ne^{-n}[-1+1] = 0$ **A**

14. $V = \frac{4}{3}\pi r^3$ so $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. $\frac{dV}{dt} = 3$ and r = 2. $3 = 16\pi \frac{dr}{dt}$. $\frac{dr}{dt} = \frac{3}{16\pi}$.

$2r = d$ so $\frac{dd}{dt} = 2\frac{dr}{dt} = \frac{3}{8\pi}$. **C**

15. Let $u = x^2$ and $dv = \cos x\ dx$. $du = 2x\ dx$ and $v = \sin x$. The integral equals $x^2 \sin x - \int_0^{\pi} 2x \sin x dx$. $u = 2x$, $dv = \sin x\ dx$, $du = 2dx$, and $v = -\cos x$.

This yields $x^2 \sin x - \left(-2x\cos x - \int_0^{\pi} -2\cos x\ dx \right) = x^2 \sin x + 2x\cos x + 2\sin x]_0^{\pi} = -2\pi$.

E

16. $x(x+2)(5-x)=10x+3x^2-x^3=y$ $\quad y'=10+6x-3x^2=0$ Solve for the positive root and plug it back into the original function to get 30.041. **D**

17. Let $u=e^x$, $dv=\cos x\ dx$, $du=e^x\ dx$, and $v=\sin x$. The integral equals

$$e^x\sin x-\int e^x\sin x\ dx=e^x\sin x-\left(-e^x\cos x-\int -e^x\cos x\ dx\right)=e^x\sin x+e^x\cos x-\int e^x\cos x\ dx$$

$$\int e^x\cos x\ dx=\frac{e^x(\cos x+\sin x)}{2}+C.$$ **A**

18. $$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2t+2}{-\sin x}\to\frac{\pi+2}{-1}=-\pi-2.$$ **A**

19.

$$\frac{dy}{d(\ln x)}=\frac{\frac{dy}{dx}}{\frac{d(\ln x)}{dx}}=\frac{3x^2+8x-4}{\frac{1}{x}}=3x^3+8x^2-4x$$ **A**

20. $\frac{1}{5-2}\int_2^5\frac{x-2}{x-1}dx$. Use $u=x-1$ and $x-2=u-1$.

$$\frac{1}{3}\int_1^4\left[1-\frac{1}{u}\right]du=\frac{x-\ln x]_1^4}{3}=1-\ln\sqrt[3]{4}.$$ **B**

21. Since $g(7)=64, f(7)=4$. $g'(x)=3[f(x)]^2 f'(x)$. $g'(7)=3\cdot16\cdot12=576$. **D**

22. $4\int\ln 2\cdot 2^x\,dx=4\cdot 2^x+C=2^{x+2}+C$. **C**

23. $y'=\frac{1}{x\ln 4}\to\frac{1}{2\ln 4}=\frac{1}{\ln 16}$. **C**

24. Let $y=u^v$ $\quad \ln y=v\ln u$ $\quad \frac{y'}{u^v}=v'\ln u+\frac{vu'}{u}$

$$y'=\frac{vu'u^v}{u}+u^v\ln uv'=vu^{v-1}\frac{du}{dx}+u^v\ln u\frac{dv}{dx}.$$ **B**

25. $\left(\frac{f(a)}{g(a)}\right)'' = \frac{[g(a)]^2(f''(a)g(a)-g''(a)f(a))-2g(a)g'(a)(g(a)f'(a)-g'(a)f(a))}{[g(a)]^4}$

$\frac{1\cdot(2\cdot1-5\cdot2)-2\cdot1\cdot2(1\cdot-1-2\cdot2)}{1}=12$. **C**

26.
$s(t)=t^4+t\cos t$
$v(t)=4t^3+\cos t-t\sin t$ **B**
$a(t)=12t^2-t\cos t-2\sin t$
$j(t)=24t+t\sin t-3\cos t \rightarrow 24\pi+3$

27. a) is not differentiable at 0, b) is not continuous at 0, and d) $f(-2)\neq f(1.5)$. c) meets all requirements. **C**

28. This is in the form $P\left(1+\frac{r}{n}\right)^{nt}$. When n approaches infinity, it becomes Pe^{rt}. Thus the answer is ae^{bc}. **D**

29. Let $y=\frac{2x\cos x\sin x}{2}=\frac{x\sin(2x)}{2}$ $y'=\frac{\sin(2x)}{2}+x\cos(2x)$
$y''=2\cos(2x)-2x\sin(2x)=2(\cos(2x)-x\sin(2x))$. **D**

30. $\int\frac{1}{4+\left(\frac{x}{3}\right)^2}dx$ $a=2$ and $u=\frac{x}{3}, du=\frac{dx}{3}$. **C**