Miami Springs Senior High School Mathematics Invitational April 16, 1994

PRE CALCULUS ANSWER KEY

1	E	495	11	B		21	D	
2	C		12	B		22	C	
3	B		13	C		23	D	
4	C		14	C		24	B	
5	E	10	15	E	12	25	C	
6	A		16	A		26	D	
7	B		17	C		27	B	
8	C		18	C		28	E	$24\sqrt{3}$
9	A		19	B		29	A	
10	D		20	A		30	C	

Precalculus Individual Test

① 5 div 1994 = 398
25 div 1994 = 79
125 div 1994 = 15
625 div 1994 = 3

$\overline{495}$ zeros $\therefore E$

② $(1-i^9) = \left[\sqrt{2}\ \text{cis}\ (-45^\circ)\right]^9$
$= 16\sqrt{2}\ \text{cis}\ (-405^\circ)$
$= 16\sqrt{2}\ \text{cis}\ (315^\circ)$
$= 16\sqrt{2}\ [\cos(315^\circ) + i\sin(315^\circ)]$
$= 16\sqrt{2}\ \left[\frac{\sqrt{2}}{2} + \frac{-\sqrt{2}}{2}i\right]$
$= 16 - 16i \quad \therefore C$

③ $103_n \div 4 = 23_n$
$n^2 + 3 = 4(2n+2)$
$n^2 - 8n - 9 = 0$
$(n-9)(n+1) = 0$
$n = 9 \quad \therefore B$

④ $$\frac{\sqrt{56+\sqrt{56+\sqrt{56\ldots}}}}{1+\cfrac{2}{1+\cfrac{2}{1+\ldots}}} = \frac{A}{B}$$

$A = \sqrt{56 + A}$ $\qquad B = 1 + \frac{2}{B}$
$A^2 = 56 + A$ $\qquad B^2 = B + 2$
$A^2 - A - 56 = 0$ $\qquad B^2 - B - 2 = 0$
$(A-8)(A+7) = 0$ $\qquad (B-2)(B+1) = 0$
$A = 8$ $\qquad B = 2$
$\frac{A}{B} = \frac{8}{2} = 4 \quad \therefore C$

⑤ $\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 6 & 3 & 1 \\ 4 & 7 & 1 \end{vmatrix}$ $\qquad \frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 6 & 3 & 1 \\ 4 & 7 & 1 \end{vmatrix}\begin{matrix} 2 & 1 \\ 6 & 3 \\ 4 & 7 \end{matrix}$

$\frac{1}{2}(20) = 10 \quad \therefore E$

⑥ $P_1(-5,3)$, $P_2(-2,-1)$ and $P_3(7,8)$
(x,y) center of gravity $= \left[\frac{(-5-2+7)}{3}, \frac{(3-1+8)}{3}\right]$
$\left(0, \frac{10}{3}\right) \quad \therefore A$

⑦ $\tan\psi = \frac{|m_1 - m_2|}{1 - m_1 m_2}$
$= \frac{|1/7 - 4/3|}{1-(1/7)(4/3)}$
$= \frac{(25/21)}{(25/21)}$
$\tan\psi = 1 \qquad \psi = 45^\circ = \frac{\pi}{4} \therefore C$

⑧ $(\sin + \cos c)^2 = \sin^2 c + \cos^2 c + 2\sin c\cos c$
$= 1 + 2\sin c\cos c$
$= 1 + \sin 2c$
$= 1 + a$
$\sin c + \cos c = \sqrt{a+1} \quad \therefore C$

⑨ $\vec{a} = 5i - 3j + k \qquad \vec{b} = -i - j - 2k$

$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 5 & -3 & 1 \\ -1 & -1 & -2 \end{vmatrix} = a \times b \quad \begin{vmatrix} i & j & k \\ 5 & -3 & 1 \\ -1 & -1 & -2 \end{vmatrix}\begin{matrix} i & j \\ 5 & -3 \\ -1 & -1 \end{matrix}$

$7i + 9j - 8k \quad \therefore A$

⑩ If $n = pq^2r^4$ is a divisor of cube C, then C must have p, q, and r as primes in its factorization. Moreover, the exponents of p, q, and r in the factorization must be multiples of 3 and they must be at least as great as 1, 2, and 4 respectively (1, 2, 4 being exponents p, q, and r in n). Thus $p^3q^3r^6 = (pqr^2)^3$ is the smallest such cube. $\therefore D$

⑪ $A = \frac{1}{2}\sqrt{s(s-a)(s-b)(s-c)}$
$AB = \sqrt{(1-0)^2 + (0-3)^2 + (2-2)^2} = \sqrt{10}$
$AC = \sqrt{(1-(-1))^2 + (0-1)^2 + (2-1)^2} = \sqrt{6}$
$BC = \sqrt{(2-(-1))^2 + (3-1)^2 + (2-1)^2} = \sqrt{6}$
$A = \frac{1}{2}\sqrt{\left(\frac{1}{2}\sqrt{10} + \sqrt{6}\right)\left(\frac{1}{2}\sqrt{10}\right)\left(\frac{1}{2}\sqrt{10}\right)\left(\sqrt{6} - \frac{1}{2}\sqrt{10}\right)}$
$A = \frac{1}{2}\sqrt{10}\sqrt{6 - \frac{1}{4}(10)}$
$A = \left(\frac{1}{2}\sqrt{10}\right)\left(\frac{1}{2}\sqrt{14}\right)$
$A = \frac{\sqrt{140}}{4} = \frac{\sqrt{35}}{2} \quad \therefore B$

⑫ $A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \quad A^3 = \begin{bmatrix} 8 & 76 \\ 0 & 27 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 1/2 & -2/3 \\ 0 & 1/3 \end{bmatrix}$

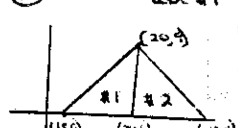
$A^3 + A^{-1} = \begin{bmatrix} 8\frac{1}{2} & 75\frac{1}{3} \\ 0 & 27\frac{1}{3} \end{bmatrix} \quad \therefore B$

⑬ A polar equation that has one of the four forms
$r = \frac{de}{1 \pm e\cos\theta} \qquad r = \frac{de}{1 \pm e\sin\theta}$
is a conic section. The conic is a parabola if $e = 1$, an ellipse if $0 < e < 1$, or a hyperbola if $e > 1$ $\therefore$ ellipse $\therefore C$

$e = 1/2$
$d = 1$
$r = \frac{\frac{1}{2}}{1 + \frac{1}{2}\cos\theta}$

⑭ $\text{Volume}_{\text{cone }\#1} = \frac{1}{3}\pi r^2 h$
$= \frac{1}{3}\pi\, 9^2(5)$
$= 135\pi$

$\text{Volume}_{\text{cone }\#2} = \frac{1}{3}\pi r^2 h$
$= \frac{1}{3}\pi\, 9^2(22)$
$= 594\pi$



$\text{Volume}_{\text{cone }\#1} + \text{Volume}_{\text{cone }\#2} = 135\pi + 594\pi$
$= 729\pi \quad \therefore C$

(15) $\frac{1}{2}\left(\frac{5!}{5}\right) = 12 \quad \therefore E$

(16) $A = \frac{2\pi}{3\pi} = \frac{2}{3} \qquad B = \frac{-6}{3\pi} = \frac{-2}{\pi} \qquad C = 3$

$$\frac{AC}{B} = \frac{(\frac{2}{3})(3)}{(\frac{-2}{\pi})} = \frac{2}{(\frac{-2}{\pi})} = -\pi \quad \therefore A$$

(17) $r = a \pm b\cos\theta$ $\qquad \frac{a}{b} < 1$ then A

$r = a \pm b\sin\theta$ $\qquad \frac{a}{b} \le 1$ then B $\qquad \frac{a}{b} = \frac{3}{2}$

$(0 < a,\ 0 < b)$ $\qquad 1 < \frac{a}{b} < 2$ then C $\qquad \therefore C$

$\frac{a}{b} \geq 2$ then D

(18) $\frac{10!}{4!\,4!\,2!} = 10\cdot9\cdot7\cdot5 = 3150\ (1)^4(1)^4(2)^2 = 12600 \quad \therefore C$

(19) $2x^5 - 11x^4 + 129x^2 - 41x + 140$

$x^5 - 11/2\, x^4 + 0x^3 + 129/2\, x^2 - 41/2\, x + 70$

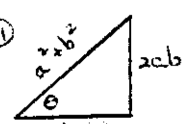
$+ \quad - \quad + \quad - \quad +$ product of the roots

$-70 \quad \therefore B$

(20) $6 + 4 + 8/3 + \ldots 6\left(\frac{2}{3}\right)^{n-1} + \ldots$

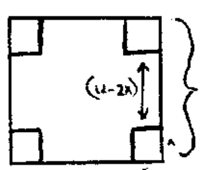
$r = 2/3 \qquad a_1 = 6$

$S = \frac{a_1}{1-r} \qquad \frac{6}{1-2/3} = 18 \quad \therefore A$

(21) 

$\sin\theta = \frac{opp}{hyp}$

$\sin\theta = \frac{2ab}{(a^2+b^2)} \quad \therefore D$

(22) 

$x(12-2x)^2 = 108$

$x[2(6-x)]^2 = 108$

$4x(6-x)^2 = 108$

$x(36-12x+x^2) = 27$

$x^3 - 12x^2 + 36x - 27 = 0$

Possible roots of this equation are factors of -27.
(1, -1, 3, -3, 9, -9, 27, and -27)

Determine which of

3	1	-12	36	-27
		3	-27	27
	1	-9	9	0

the integral answer is $x = 3 \quad \therefore C$

(23) $t = kxy$ $\quad$ K is a nonzero constant. $\quad f(x)$ = BACTERIA per m. $\quad x$ = bacteria present

$60 = k(1000)(1{,}000{,}000 - 1000)$

$k = \frac{60}{999{,}000{,}000}$

$k = \frac{1}{16{,}650{,}000}$

$f(x) = \frac{x(1{,}000{,}000 - x)}{16{,}650{,}000}$

$f(100{,}000) = \frac{100{,}000(1{,}000{,}000 - 100{,}000)}{16{,}650{,}000}$

$= \frac{100{,}000(900{,}000)}{16{,}650{,}000}$

* 5405.405405

∴ the rate of growth is 5405 bacteria per minute. ∴ D

* Note Calculator overflow seems to cause 540!!

(24) $r^2 - 8r\left(\frac{x}{r}\right) + 6r\left(\frac{y}{r}\right) = 0$

$x^2 + y^2 - 8x + 6y = 0$

$(x-4)^2 + (y+3)^2 = 25$

radius = 5

$25\pi \quad \therefore B$

(25) $16x^2 + \quad + 64x - 18y - 71 = 0$

$16x^2 + 64x + 9y^2 - 18y = 71$

$16(x^2+4x+4) + 9(y^2-2y+1) = 71+64+9$

$\frac{(x+2)^2}{9} + \frac{(y-1)^2}{16} = 1$

base $= ab\pi = 12\pi$

$V_{cone} = 1/3\ Bh$

$1/3\,(12\pi)10$

$40\pi \quad \therefore C$

(26) Let the third root of the given equation be S. the sum of the roots is zero, $2a + S = 0$; $S = -2a$
The sum of the products of the roots taken two at a time is 8.

$(a+bi)(a-bi) + (a+bi)s + (a-bi)s = 8$

$(a^2+b^2) + a(-2a) + bi(-2a) + a(-2a) - bi(2a)$

$b^2 - 3a^2 = 8 \quad \therefore D$

(27) Simplify:

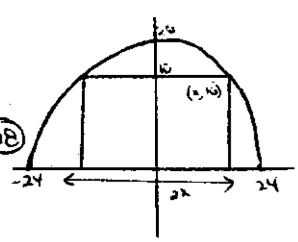
$(1+\sec\theta)(1-\cos\theta) = 1 + \sec\theta - \cos\theta - \sec\theta\cos\theta$

$= 1 + \frac{1}{\cos\theta} - \cos\theta - 1$

$= \frac{1-\cos^2\theta}{\cos\theta}$

$= \frac{\sin^2\theta}{\cos\theta}$

$= \tan\theta\sin\theta \quad \therefore B$

(28) 

$\frac{x^2}{a} + \frac{y^2}{b} = 1$

Let $a = 24$ & $b = 20$

$\frac{x^2}{576} + \frac{y^2}{400} = 1$

$(x, 10)$ is in the graph

$\frac{x^2}{576} + \frac{100}{400} = 1$

$x^2 = 432 \to x = 12\sqrt{3}$

The width of arch is $24\sqrt{3} \quad \therefore E$

(29) $i^{250-50+5+8} = i^{213}$

$i^{212} = 1$

$1 \cdot i = i \quad \therefore A$

(30) $(\bar{p}+\bar{q})q + q$

$(q\bar{p} + \bar{q}q + q) \to q\bar{p} + q \to q(\bar{p}+1)$

$q \rightarrow \quad \therefore C$