SCHOOL MATH BOWL

Alpha

HAWAII 1993

1. If Arc denotes principal value and if $x = \text{Arcsin}(0.6) + \text{Arccos}\left(\frac{15}{17}\right)$, find the value of sinx.

Let $\alpha = \text{Arcsin}(0.6)$

$\beta = \text{Arccos}\left(\frac{15}{17}\right)$

Then $\sin x = \sin(\alpha + \beta)$

$= \sin\alpha \cos\beta + \cos\alpha \sin\beta$

$= \frac{3}{5} \cdot \frac{15}{17} + \frac{4}{5} \cdot \frac{8}{17}$

$= \boxed{\frac{77}{85}}$

2. An isosceles triangle has two sides - each with length 4. Find the length of the third side that gives the maximum area of the isosceles triangle.

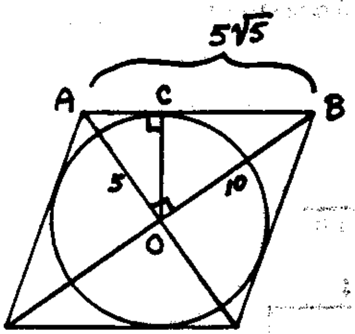
$A_\Delta = \frac{1}{2} a b \sin C$

$= \frac{1}{2}(4)(4)\sin C$

so the area of the triangle is a maximum when sin C is a maximum. Sin C is a maximum when $C = 90°$.

$\therefore$ the third side is $\boxed{4\sqrt{2}.}$

3. The diagonals of a rhombus are 10 and 20. Find, in simplified radical form, the radius of the circle inscribed in the rhombus.



Since the diagonals of a rhombus are $\perp$ bisectors of each other, $AO = 5$ + $BO = 10$.

By the Pythagorean Theorem, $AB = 5\sqrt{5}$.

Since the area of ΔAOB is $\frac{1}{2}(AO)(BO) = \frac{1}{2}(AB)(CO)$

$5(10) = 5\sqrt{5}\,(CO)$

$\therefore CO = \frac{10}{\sqrt{5}} = \frac{10\sqrt{5}}{5} = \boxed{2\sqrt{5}}$

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4. Give the exact value of this infinite continued fraction: $3+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\dots}}}$

$x = 3 + \dfrac{1}{2+\frac{1}{x}} = 3 + \dfrac{1}{\frac{2x+1}{x}} = 3 + \dfrac{x}{2x+1}$

$= \dfrac{7x+3}{2x+1}$

Thus $2x^2 + x = 7x + 3$

$2x^2 - 6x - 3 = 0$

So $x = \dfrac{6 \pm \sqrt{36-4(2)(-3)}}{2(2)}$

$= \dfrac{3 \pm \sqrt{15}}{2}$

We want the positive root: $\boxed{\dfrac{3+\sqrt{15}}{2}}$

5. How many triangles can be inscribed inside a convex pentagon so that each vertex of each triangle touches a vertex of the pentagon?

A triangle is determined by a choice of 3 vertices. (Notice that any choice is possible.)

There are 5 positions to choose from, so there are $\binom{5}{3} = \boxed{10 \text{ possible ways}}$

6. Everyone in a class of 30 students scored less than 6 points out of 100 on an exam. The average score was $3.\bar{6}$ (where the 6 repeats), and exactly 90% of the class scored above average. There are 4 possibilities for the scores of the 30 students. Fill in the chart below by indicating the number of students making each score for each possibility. Zeroes need not be placed on your answer for the chart.

Since 90% of 30 is 27, at least 27 students must have scored 4 or more.

The possibilities are:

Score	Possibility #1	Possibility #2	Possibility #3	Possibility #4
5	2	1		
4	25	26	27	27
3				
2			1	
1		1		2
0	3	2	2	1

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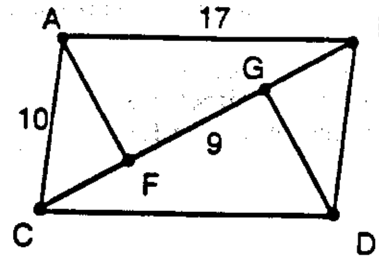
HAWAII 1993

7. Given $\overline{AF} \perp \overline{CB}$, $\overline{DG} \perp \overline{BC}$, AB = 17, AC= 10, and FG = 9, compute the area of parallelogram $\square$ABDC:



$\triangle AFC \cong \triangle DGB$

Let $AF = y$ and $CF = x$. Then $BG = x$.

By the Pythagorean Theorem:

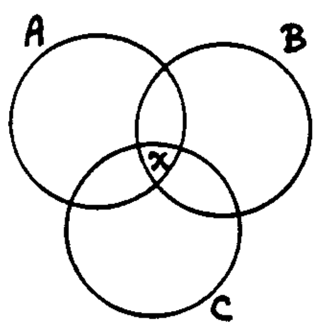
$$\begin{cases} x^2 + y^2 = 10^2 \\ (9+x)^2 + y^2 = 17^2 \end{cases}$$

Expanding the second equation and subtracting the first gives:

$18x + 81 = 189$ so $x = 6$. Then $y = 8$.

The area of the parallelogram is $CB \cdot AF = (2x+9)y = 21 \cdot 8 = \boxed{168}$

8. Given three sets of objects A, B, and C. One-fifth of the objects in A are in both B and C; $\frac{2}{3}$ of the objects in B are in both A and C; $\frac{1}{4}$ of the objects in C are in both A and B. What are the relative numbers of objects in A, B, and C? (Express the relative numbers as the ratio A:B:C with the smallest possible integers.)



From the given:

$$x = \frac{A}{5} = \frac{2B}{3} = \frac{C}{4}$$

$3A = 10B$ or $\frac{A}{B} = \frac{10}{3}$

$4A = 5C$ or $\frac{A}{C} = \frac{5}{4} = \frac{10}{8}$

$8B = 3C$ or $\frac{B}{C} = \frac{3}{8}$

$\therefore A : B : C = \boxed{10 : 3 : 8}$

9. If x and y are integers such that $x^5 = 1\bullet,\bullet\bullet\bullet,\bullet\bullet8$ and $y^5 = 5\bullet\bullet,\bullet\bullet4$, where $\bullet$ respresents a missing digit, compute $\frac{x^5}{y^5}$.

Since $x^5 \equiv x \mod 10$, the final digit of x must be 8.

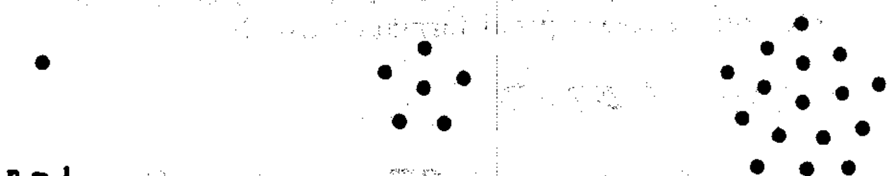
Since $20^5 = 3{,}200{,}000$ and $30^5 = 24{,}300{,}000$, x must be 28.

Similarly $y = 14$.

Thus $\frac{x^5}{y^5} = \frac{28^5}{14^5} = 2^5 = \boxed{32}$

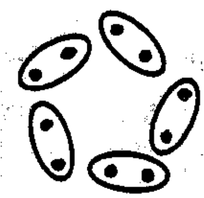
10. Consider the pentagonal-shaped patterns:

Compute the number of dots on the perimeter of the pattern with n = 100.



n = 1 n = 2 n = 3

By grouping the points on the perimeter as follows:



one can see that the number of points on the perimeter is $5(n-1)$.

For $n = 100$, this is $\boxed{495.}$

11. If a parabola $y = ax^2+bx+c$ goes through the points (1,0), (–1,6), and (2,3), what is its slope at $x = 3$?

This gives a system of 3 equations:

$$0 = a(1)^2 + b(1) + c$$
$$6 = a(-1)^2 + b(-1) + c$$
$$3 = a(2)^2 + b(2) + c$$

$$0 = a + b + c$$
$$6 = a - b + c$$
$$3 = 4a + 2b + c$$

Gaussian elimination gives: $a + b + c = 0$
$b = -3$
$c = 1$
$a = 2$

Thus $y = 2x^2 - 3x + 1$.

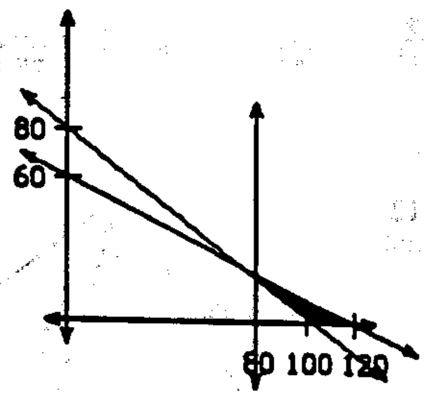
Therefore the slope at $x=3$ is $y' = 4(3) - 3 = \boxed{9}$

12. K-mart sells men's bathing suits for \$12 each and rain-ponchos for \$15 each. They can fit twice as many bathing suits as ponchos on a rack and there is space for 120 bathing suits (or 60 ponchos, etc.) They must make at least \$1200 per day in total revenue on these two items to cover their expenses. They need to sell at least 80 bathing suits per day. Assuming that everything on the shelves will sell, how should they stock the shelves to sell as many ponchos as possible?

If x = # of bathing suits and y = # of ponchos, the constraints are:

Shelf space:	$x + 2y \le 120$
Gross Income:	$12x + 15y \ge 1200$
Sales Target:	$x \ge 80$
Positivity:	$y \ge 0$

This region looks like:



where the point of intersection of the oblique lines is to the left of the vertical line. Thus the maximum y occurs at the intersection of $x = 80$ and $x + 2y = 120$, which is at $x = 80$ + $y = 20$. Hence there are $\boxed{\text{80 bathing suits + 20 ponchos.}}$

13. If $f(x) = x^3 + ax^2 + bx$ has an inflection point at $x = 3$, and the horizontal distance between its local maximum and local minimum is 4, what is $f(1)$?

Since $f''(x) = 6x + 2a$, we know that $0 = 6(3) + 2a$ or $a = -9$

The local extrema are the zeroes of $f'(x) = 3x^2 + 2ax + b$

$= 3x^2 - 18x + b$ which are:

$$2\left(\frac{\sqrt{(-18)^2 - 4(3)b}}{2(3)}\right) = 4$$

units apart.

Thus $(12)^2 = (-18)^2 - 12b$

so $b = \dfrac{(18)^2 - (12)^2}{12} = 3(9) - 12 = 15$

Therefore, $f(1) = 1 + a + b = 1 + (-9) + (15) = 7$

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14. What is the probability of rolling a number larger than a 4 on a normal 6-sided die at most 3 out of 5 times?

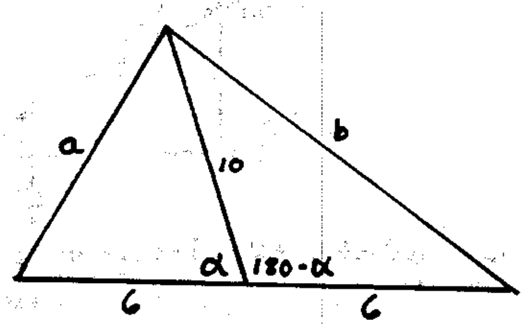
Since the probability of rolling more than a 4 is $\frac{2}{6} = \frac{1}{3}$, using the Binomial formula gives the probability of rolling this at most 3 out of 5 times is:

$$\sum_{i=0}^{3} \binom{5}{i}\left(\frac{1}{3}\right)^{i}\left(\frac{2}{3}\right)^{5-i} = 1 - \sum_{i=4}^{5} \binom{5}{i}\left(\frac{1}{3}\right)^{i}\left(\frac{2}{3}\right)^{5-i} = 1 - \binom{5}{4}\left(\frac{1}{3}\right)^{4}\left(\frac{2}{3}\right)^{1} - \binom{5}{5}\left(\frac{1}{3}\right)^{5}\left(\frac{2}{3}\right)^{0}$$

$$= 1 - 5\left(\frac{2}{3^5}\right) - \frac{1}{3^5} = 1 - \frac{11}{243} = \boxed{\frac{232}{243}}$$

15. The sides of a triangle are a, b, and 12.
The median to the side of 12 has length 10.

Find: $a^2 + b^2$



By the law of cosines:

1 $\quad a^2 = 6^2 + 10^2 - 2 \cdot 6 \cdot 10 \cos\alpha$

$\quad b^2 = 6^2 + 10^2 - 2 \cdot 6 \cdot 10 \cos(180 - \alpha)$

But $\cos(180 - \alpha) = -\cos\alpha$

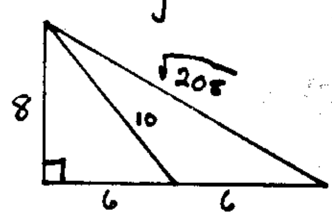
3 $\quad b^2 = 6^2 + 10^2 + 2 \cdot 6 \cdot 10 \cos\alpha$

Adding equations 1 and 2 gives:

$$a^2 + b^2 = 2(6^2 + 10^2) = \boxed{272}$$

Note: you could also use a special case:

ex:



$$8^2 + \left(\sqrt{208}\right)^2 = \boxed{272}$$

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16. Sum the numbers in the following rectangular array:

$$
\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & \cdot & \cdot & \cdot & 97 & 98 & 99 & 100 \\
2 & 3 & 4 & 5 & \cdot & \cdot & \cdot & 98 & 99 & 100 & 101 \\
3 & 4 & 5 & 6 & \cdot & \cdot & \cdot & 99 & 100 & 101 & 102 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
98 & 99 & 100 & 101 & \cdot & \cdot & \cdot & 194 & 195 & 196 & 197 \\
99 & 100 & 101 & 102 & \cdot & \cdot & \cdot & 195 & 196 & 197 & 198 \\
100 & 101 & 102 & 103 & \cdot & \cdot & \cdot & 196 & 197 & 198 & 199
\end{array}
$$

The first column sums to $\frac{100(1+100)}{2}$

The second column sums to $\frac{100(2+101)}{2}$, etc.

Thus the total is $50(101 + 103 + \ldots + 299)$

$$= 50\left[\frac{100(101+299)}{2}\right]$$

$$= 50^2(400)$$

$$= 100^3$$

$$= \boxed{10^6 \text{ or } 1{,}000{,}000}$$