1992 National Mu Alpha Theta Convention

Alpha Bowl Answers:

1. $\frac{\pi}{4}, \frac{9\pi}{4}$
2. All m > 0
3. x = 0, 16
4. (1,1), (1,-1), (4,-2)
5. S = 9/2
6. $\frac{25}{7776}$
7. $y = \frac{2^{x+1}+1}{2}$
8. 5
9. 100 ln10 - 90
10. $|x| > 1$ or $x < -1 \cup x > 1$
11. y = 1/2
12. -112 ft/sec.
13. S = 5
14. x = $294
15. SA = $4\sqrt{3}$

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Alpha Bowl Solutions:

1. $\frac{1}{2}\sin 2\theta$ must be positive and $\sin\theta$ must also be positive.

   $\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta = \sin^2\theta$ implies that $\sin\theta\left(\cos\theta - \sin\theta\right) = 0$.

   Since, $\sin\theta > 0$, $\cos\theta = \sin\theta$, *or* $\tan\theta = 1$. Thus, the values sought are $\frac{\pi}{4}, \frac{9\pi}{4}$.

2. The quadratic is the eqn. of a circle: $\left(x-2\right)^2 + \left(y-3\right)^2 = m$. Since, the slope of the tangent is -1, the eqn. of the radius through the tangent pt. is y -3 = x - 2 or x - y = -1. Solving x - y = -1 and x + y = 5 + $\sqrt{2m}$ give x = 2 + $\sqrt{2m}/2$ and y = 3 + $\sqrt{2m}/2$. Substituting into the circle eqn., we get $m = m$, which is true for all $m > 0$.

3. Factoring gives: $x^{1/2}\left(2 + x^{1/4} - x^{2/4}\right) = 0 \Rightarrow x^{1/2}\left(2 - x^{1/4}\right)\left(1 + x^{1/4}\right) = 0$ and x = 0, 16.

4. Since $x^{x+y} = y^4$ and $y^{x+y} = x$, $\left(y^{x+y}\right)^{x+y} = y^4$ or $y^{(x+y)^2} = y^4$. Four possibilities exist: (1) y = 0, but not possible in the 1st eqn.; (2) y = 1, which implies x= 1.; (3) y = -1, which also implies x = 1; (4) for any other y, $(x + y)^2 = 4$ so x + y = ±2. If x + y = 2 then $x = y^2$, so $y^2 + y = 2$ and y = -2 or 1. If y = -2 then x = 4. If x + y = -2, then x would be a fraction. So, the only ordered pairs of integer solutions are: (1,1), (1,-1), and (4, -2).

5. Let S be the sum. If $S = 1 + \frac{4}{3} + \frac{9}{3^2} + \frac{16}{3^3} + \ldots$, then $3S = 3 + 4 + \frac{9}{3} + \frac{16}{3^2} + \ldots$

   Subtracting, we get $2S = 6 + \frac{5}{3} + \frac{7}{3^2} + \frac{99}{3^3} + \ldots$

   So, $6S = 18 + 5 + \frac{7}{3} + \frac{9}{3^2} + \ldots and 6S - 2S = 17 + \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \ldots$

   $= 17 + \frac{2/3}{1 - 1/3} = 17 + 1.$

   Thus, 4S = 18 and S = 9/2.

6. For 6 different rolls, the dice will add up to 7. Thus, the probability of a 7 on a single toss is 1/6. The probability that 7 will not show is 1 - 1/6 = 5/6. So, the answer is:
$\frac{1}{6} \bullet \frac{1}{6} \bullet \frac{1}{6} \bullet \frac{5}{6} \bullet \frac{5}{6} = \underline{\frac{25}{7776}}$

7. The function $y = \log_2 \frac{2x-1}{2}$ becomes $x = \log_2 \frac{2y-1}{2}$

$$2^x = \frac{2y-1}{2} \Rightarrow \underline{y = \frac{2^{x+1}+1}{2}}$$

8. $\lim_{x\to 2} 8x - 11 = 5, \ \lim_{x\to 2} 2x^2 - 3 = 5$

$$\lim_{x\to 2} 8x - 11 \le \lim_{x\to 2} f(x) \le \lim_{x\to 2} 2x^2 - 3$$

$$5 \le \lim_{x\to 2} f(x) \le 5 \Rightarrow \underline{\lim_{x\to 2} f(x) = 5}$$

9. f '(t) = $10^{t+1}\ln 10$ and f '(1) = 100 ln10.
Avg. rate of growth is $\frac{f(1)-f(0)}{1-0} = 90$.
So, f '(1) - Avg. rate = <u>(100 ln10 - 90).</u>

10. $f'(x) = (x^2-1)e^{\frac{x^3}{3}-x}$, $e^{\frac{x^3}{3}-x} > 0$ for all real x. So, $x^2 - 1 > 0$ when $\underline{|x| > 1}$.

11. $\lim_{x\to +\infty} \frac{|x|}{|x|+x} = \lim_{x\to +\infty} \frac{x}{x+x} = \frac{1}{2}$ and $\lim_{x\to -\infty} \frac{|x|}{|x|+x} = \lim_{x\to -\infty} \frac{-x}{-x+x} = -\infty$

So, the only horiz. asy. is $\underline{y = \frac{1}{2}}$

12. The time of impact occurs when $s(t) = 0$ for the second time.
$16t(7 - t) = 0$ at $t = 0$, $t = 7$. The velocity at $t = 7$ is $s'(7) =$ -112 ft/sec.

13. Note that $\sin^2 10° + \sin^2 80° = 1$, and $\sin^2 20° + \sin^2 70° = 1, \ldots$
and finally $\sin^2 90° = 1$. Thus, $S = 5$.

14. The oldest contributed one-third the total cost. The 2nd oldest contributed one-fourth the total, the third contributed one-fifth the total, the fourth contributed one-sixth the total and the youngest contributed \$14.70. If x = cost, then x/3 +x/4 +x/5 +x/6 + \$14.70 = x and x = \$294.

15. The solid is an octahedron, with each face and equilateral triangle of side-length $\sqrt{2}$.
The surface area is therefore $4\sqrt{3}$.